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A METHOD FOR THE SOLUTION OF NORMAL EQUATIONS

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INTRODUCTION

The methods employed for the solution of a set of normal equations have been modified in recent years to employ modern calculating machines to the fullest extent. Remarkable among these modifications is the abbreviated Doolittle solution due to Dwyer (2), which requires a considerably reduced number of entries in the table of solution of such a system of equations. Dwyer (2, p. 443) claims that this method "is unquestionably the method which should be used by those seeking numerical solutions to the classical problems of multiple and partial correlation." Hotelling's experience with this and other methods has led him to state (4, p. 16) that "for the mass of least-square and other problems in which the inverse of a matrix is needed, the best procedure appears to begin with one of the methods described by Dwyer," etc. It is the purpose of this article to call attention to a method which is simpler, at least for relatively small numbers of constants to be fitted, than Dwyer's abbreviation of the Doolittle method.

THEORY OF THE METHOD

The method is based on Laplace's development of a determinant, stated by Bôcher (1, p. 26) as follows:

"Pick out any m rows (or columns) from a determinant D, and form all the m-rowed determinants from this matrix. The sum of the products of each of these minors by its algebraic complement is the value of D."

The application of this principle to the solution of a set of normal equations will be explained now.

Let

$$A_{123\cdots n} = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & m_1 \\ a_2 & b_2 & c_2 & \cdots & m_2 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_n & b_n & c_n & \cdots & m_n \end{vmatrix}$$

, and let, for $i < j < k$, and with reference to $A_{123\cdots n}$,

$$A_{ij} = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} = a_ib_j - a_jb_i,$$

for example, $$A_{25} = \begin{vmatrix} a_2 & b_2 \\ a_5 & b_5 \end{vmatrix} = a_2b_5 - a_5b_2 ;$$

$$B_{ij} = \begin{vmatrix} b_i & c_i \\ b_j & c_j \end{vmatrix} = b_ic_j - b_jc_i ,$$

for example, $$B_{24} = \begin{vmatrix} b_2 & c_2 \\ b_4 & c_4 \end{vmatrix} = b_2c_4 - b_4c_2 ;$$

$$A_{ijk} = \begin{vmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{vmatrix} ; \quad \text{for example,} \quad A_{146} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_4 & b_4 & c_4 \\ a_6 & b_6 & c_6 \end{vmatrix} ;$$

$$A_{ioj} = \begin{vmatrix} a_i & c_i \\ a_j & c_j \end{vmatrix} = a_ic_j - a_jc_i ,$$

for example, $$A_{206} = \begin{vmatrix} a_2 & c_2 \\ a_6 & c_6 \end{vmatrix} = a_2c_6 - a_6c_2 \text{ ; etc.}$$

Now, the application of the above-mentioned principle will show that

$$A_{123} = A_{12}c_3 - A_{13}c_2 + A_{23}c_1$$

$$A_{1234} = A_{123}d_4 - A_{124}d_3 + A_{134}d_2 - A_{234}d_1 \text{ , etc.}$$

The sign of any given term will be positive if "0" or an even number of inversions of adjacent subscripts is required to bring them into an ascending order, and negative if said number of inversions is odd.

Fisher's (3, p. 144) modification of Gauss' (5, p. 60 and 84) method of undetermined multipliers applied to the set of normal equations:

$$\begin{aligned} Aa_1 + Bb_1 + \cdots + Mm_1 &= Say, \\ Aa_2 + Bb_2 + \cdots + Mm_2 &= Sby, \\ \vdots \qquad \vdots \qquad\quad & \vdots \end{aligned}$$

and $$Aa_n + Bb_n + \cdots + Mm_n = Smy,$$

where $a_1 = Sa^2$, $b_1 = a_2 = Sab$, $\cdots$, $b_2 = Sb^2$, $c_2 = b_3 = Sbc$; etc., requires the solution of the following sets of equations for C_{aa} , C_{ab} , $\cdots$, C_{bb} , C_{bc} , $\cdots$, and C_{mm} :

$$\text{1st set:} \quad \begin{aligned} C_{aa}a_1 + C_{ab}b_1 + \cdots + C_{am}m_1 &= 1 \\ C_{aa}a_2 + C_{ab}b_2 + \cdots + C_{am}m_2 &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad & \vdots \\ C_{aa}a_n + C_{ab}b_n + \cdots + C_{am}m_n &= 0 \end{aligned}$$

$$
\begin{aligned}
\text{2nd set: } & C_{ab}a_1 + C_{bb}b_1 + \cdots + C_{bm}m_1 = 0\\
& C_{ab}a_2 + C_{bb}b_2 + \cdots + C_{bm}m_2 = 1\\
& \quad \vdots \\
& C_{ab}a_n + C_{bb}b_n + \cdots + C_{bm}m_n = 0\\
n\text{th set: } & C_{am}a_1 + C_{bm}b_1 + \cdots + C_{mm}m_1 = 0\\
& C_{am}a_2 + C_{bm}b_2 + \cdots + C_{mm}m_2 = 0\\
& \quad \vdots \\
& C_{am}a_n + C_{bm}b_n + \cdots + C_{mm}m_n = 1.
\end{aligned}
$$

If one now lets $R = 1/A_{1234\cdots n}$, one can obtain the following relations for the c_{ij}'s.

From the 1st set of equations, one obtains

$$
\begin{aligned}
C_{aa} &= B_{23\cdots n}R,\\
C_{ab} &= -A_{2034\cdots n}R,\\
C_{ac} &= A_{2304\cdots n}R,\\
&\ \vdots\\
C_{am} &= \pm A_{234\cdots n}R;
\end{aligned}
$$

from the 2nd set, one obtains similarly,

$$
\begin{aligned}
C_{bb} &= A_{1034\cdots n}R\\
C_{bc} &= -A_{1304\cdots n}R,\\
&\ \vdots\\
C_{bm} &= \pm_{134\cdots n}R;
\end{aligned}
$$

and finally, from the nth set,

$$C_{mm} = A_{123\cdots(n-1)}R.$$

The values of the c_{ij}'s may then be found by obtaining the products of a number of determinants of the order $(n - 1)$ by the reciprocal of $A_{123\cdots n}$, which itself may be evaluated from the relation

$$A_{123\cdots n} = A_{123\cdots(n-1)}M_n - A_{123\cdots(n-2)n}M_{n-1} + \cdots \pm A_{234\cdots n}M_1,$$

the last term being positive or negative according to the respective number of inversions of subscripts required as mentioned above. The determinants which are coefficients of the m's in the last expression are the coefficients of R in the relations for C_{mm}, $C_{(m-1)m}$, $\cdots$, C_{am} above.

APPLICATION OF THE METHOD

The application of the method will be described in connection with the solution of a set of 3 normal equations involving 3 predictors.

Normal equations:

$$Aa_1 + Bb_1 + Cc_1 = Say,$$

$$Aa_2 + Bb_2 + Cc_2 = Sby,$$

$$Aa_3 + Bb_3 + Cc_3 = Scy.$$

From the previous relations,

$$C_{aa} = B_{23}R = (b_2c_3 - b_3c_2)R,$$

$$C_{ab} = -A_{203}R = -(a_2c_3 - a_3c_2)R,$$

$$C_{ac} = A_{23}R = (a_2b_3 - a_3b_2)R,$$

$$C_{bb} = A_{103}R = (a_1c_3 - a_3c_1)R,$$

$$C_{bc} = -A_{13}R = -(a_1b_3 - a_3b_1)R,$$

$$C_{cc} = A_{12}R = (a_1b_2 - a_2b_1)R,$$

$$A_{123} = A_{12}c_3 - A_{13}c_2 + A_{23}c_1,$$

and $R = 1/A_{123}$.

To evaluate the c_{ij}'s, therefore, one needs to know A_{12}, A_{13}, A_{103}, A_{23}, A_{203}, and B_{23} to start with. The values of these expressions, which consist of differences between products of two terms each, as $pq - rs$, may be obtained from the machine without recording any intermediate figures. From these values and the original data, A_{123} and its reciprocal, R, may be evaluated, and from these, by setting R in the machine, the values of the c_{ij}'s directly. The values of A, B and C and their standard errors may then be obtained in the usual way.

The application of the procedure outlined above to the same data from Carver used by Dwyer (2) to illustrate his method follows.

$a_1 = 1.000$	$b_1 = 0.313$	$c_1 = 0.280$	$Say = 0.495$
$a_2 = 0.313$	$b_2 = 1.000$	$c_2 = 0.652$	$Sby = 0.650$
$a_3 = 0.280$	$b_3 = 0.652$	$c_3 = 1.000$	$Scy = 0.803$
			$Syy = 1.000$

$$
\begin{array}{llll}
A_{12} = a_1b_2 - a_2b_1 = & 0.902031 & C_{cc} = A_{12}R = & 1.758997 \\
A_{13} = a_1b_3 - a_3b_1 = & 0.564360 & C_{bc} = -A_{13}R = & -1.100525 \\
A_{103} = a_1c_3 - a_3c_1 = & 0.921600 & C_{bb} = A_{103}R = & 1.797157 \\
A_{23} = a_2b_3 - a_3b_2 = & -0.075924 & C_{ac} = A_{23}R = & -0.148055 \\
A_{203} = a_2c_3 - a_3c_2 = & 0.130440 & C_{ab} = -A_{203}R = & -0.254363 \\
B_{23} = b_2c_3 - b_3c_2 = & 0.574896 & C_{aa} = B_{23}R = & 1.121070
\end{array}
$$

$$A_{123} = A_{12}c_3 - A_{13}c_2 + A_{23}c_1 = 0.512810$$

$$R = 1/A_{123} = 1.950040$$

$$A = C_{aa}Say + C_{ab}Sby + C_{ac}Scy = 0.270706$$

$$B = C_{ab}Say + C_{bb}Sby + C_{bc}Scy = 0.158521$$

$$C = C_{ac}Say + C_{bc}Sby + C_{cc}Scy = 0.623846$$

Reduced $Syy = 1 - ASay - BSby - CScy = 0.262014$.

The information contained in rows 4 to 10 inclusive, of table VII, of Dwyer's article (2), which required *37* entries (disregarding the zeros), is found in the last *12* rows of the preceding scheme, which contains but *18* entries. In addition, the preceeding scheme contains the values of C_{ac}, C_{ab}, and C_{bc}, which do not appear in said table VII, but are essential for the statistical evaluation of differences between A, B, and C. The calculation of these values would require still 3 more entries in that table, raising the total number of entries to 40.

For the case of 4 normal equations with 4 predictors the respective numbers of entries would be *63* in the abbreviated Doolittle solution method and *40* in the proposed method. Two additional advantages of the proposed method as against Dwyer's abbreviated Doolittle solution are that the operations tend to be simpler and errors tend to be of less importance in the proposed method, since an error will not invalidate such a large proportion of the work performed subsequently to it as it does in the Doolittle technique when no check columns are included.

SUMMARY

A new method of solving normal equations is presented. A comparison is made of the application of the method to a multiple regression case used by Dwyer to illustrate his abbreviation of the Doolittle method. The proposed method for relatively small numbers of unknowns at least, is simpler than Dwyer's method, considered to date to be one, if not the most efficient for the purpose.

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