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A taxonomy for models used in developing number sense

Una taxonomía para los modelos utilizados en el desarrollo del sentido numérico

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Abstract

Number sense refers to a collection of skills that play a central role in mathematics education at all school levels. Its study has centered mostly on numerical issues related to primary school, while there is no consensus about its importance in mathematics education beyond middle school. Our hypothesis is that the study of number sense must include much more than numbers, their relations and operations, and that it is essential for students to be fully prepared to undertake mathematics courses at the university level. Students must develop, at school level, robust mental models for the various nuances of "number", such as: magnitudes, number systems, estimation and measurement processes, as well as the algebraic, variational and probabilistic dimensions of number. We propose a taxonomy for the models useful in developing number sense, and suggest ideas on how teachers can use it to translate those models into mental structures to help students build the basis for the study of mathematics in higher education.

Keywords: number sense, real numbers, dimensions of number sense, numbers

Resumen
El sentido numérico se refiere a una colección de habilidades que desempeñan un papel central en la educación matemática en todos los niveles escolares. Su estudio se ha centrado principalmente en cuestiones numéricas relacionadas con la escuela primaria, y no hay consenso sobre su importancia en la educación matemática más allá de la escuela intermedia. Nuestra hipótesis afirma que el sentido numérico debe incluir mucho más que números, sus relaciones y operaciones, y que es esencial para que los estudiantes estén completamente preparados para emprender cursos de matemáticas a nivel universitario. Los alumnos deben desarrollar, a nivel escolar, modelos mentales robustos para las diversas acepciones de "número", tales como: las magnitudes, los sistemas numéricos, los procesos de medición y estimación, y las dimensiones algebraicas, variacionales y probabilísticas del número. Proponemos una taxonomía de los modelos útiles para desarrollar el sentido numérico y sugerimos ideas sobre cómo los maestros pueden usarla para traducir dichos modelos en estructuras mentales que ayuden a los estudiantes a construir la base para el estudio de las matemáticas en educación superior.

Palabras clave: sentido numérico, pensamiento numérico, numeración, didáctica de los números

Sandow: Ramanju, they all call you a genius.
Ramanujan: What! Me a genius! Look at my elbow, it will tell you the story.
Sandow: What is all this Ramanju? Why is it so rough and black?
Ramanujan: My elbow has become rough and black in making a genius of me! Night and day I do my calculation on slate. It is too slow to look for a rag to wipe it out with. I wipe out the slate almost every few minutes with my elbow.
Ramanujan – The Man and the Mathematician (pp. 25-26).

Antecedents

In February 1989, the National Science Foundation sponsored a conference, convened in San Diego, California, with the purpose of establishing the foundations for research on number sense and related topics. The discussions were focused on the following questions:

- What is number sense? How do we assess it? How do we teach it? How is it linked to mental computation and computational estimation?
- What research questions regarding these issues need to be addressed? What are the theoretical foundations for this research? How does research in other areas of mathematics learning relate to this agenda and to the foundations for this research?

Participants, mostly researchers from psychology and mathematics education, were unable to completely resolve either of these issues. However, the conference proceedings contain valuable comments and articles that shed light on the questions raised. The first four papers were written by psychologists; the rest were authored by mathematics educators whose primary research during the previous decade was related to number sense (mainly on estimation and mental computation). The latter group had a "larger stake in the conference" since they were seeking theoretical models that would "help prioritize research questions and guide them in designing studies" (Sowder & Schapelle, 1989, p. 3).

The interactions of the two groups of participants in the conference revealed a tension between the mathematics education researchers, who believe that number sense could be studied using their fields' customary research methods, and the psychologists, who insist that a new "multiple dimensional perspective" was needed for that purpose, such as the one used to investigate intelligence (Resnick, 1989). In other words, the usual methods used in mathematics education research were claimed to be inadequate for the study of number sense, which is more concerned with the acquisition of "pieces" of mathematical knowledge rather than with the development of links between those pieces of knowledge (Marshall, 1989, p. 2). The call of number sense and estimation researchers for theoretical frameworks to their research ended in disappointment, and this fact was expressly stated in some of the papers submitted for the proceedings. Among the voiced regrets was the inadequate attention the conference gave to the topics of estimation and mental computation, as the discussion consistently drifted towards number sense as a conceptual construct, and away from the topics of central interest for research in mathematics education.

That same year, the Mathematical Association of America (MAA) established a subcommittee on Quantitative Literacy with the purpose of establishing, among other things, how university students can use mathematics in the solution of problems related to their field of study. This was the starting point for a series of activities aimed at defining the "quantitative literacy" construct and specifying the contents and processes to be studied in universities, as well as the competences students should obtain, and how to evaluate them. In 2001 the National Council on Education and the Disciplines (NCED) sponsored a national forum held at the National Academy of Sciences in order to answer the question: *Why quantitative literacy matters in schools and universities?*

Steen (2001) was one of the pioneers in trying to determine the quantitative and mathematical requirements a person should have for contemporary work and responsible citizenship. He defined quantitative literacy as the combination of arithmetic with the complex logical reasoning used to solve the issues that modern society faces (Steen, 2004). He suggested the contents of arithmetic, statistics, modeling and probability as fundamental, that together, with reasoning, data analysis and computer use would be fundamental for the training of the responsible citizens.

While the MAA dealt with the theoretical and curricular aspects of quantitative literacy, the Organization for Economic Co-operation and Development (OECD) was engaged in its assessment. To develop its test battery for the Programme for International Student Assessment (PISA), it defined quantitative literacy as "an individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognize the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens" (OECD, 2012, p. 4). The OECD indicates quantitative literacy consists of four phenomenological categories: quantity, space and form, relationships, and uncertainty (OECD, 2012).

De Lange (2003) emphasizes the importance of the phenomenological categories identified by the OECD as constituents of quantitative literacy. He also indicates that mathematical concepts must be learned through problem solving in appropriate environments and with opportunities for progressive mathematization and generalization. In addition, he proposes that the longitudinal development of mathematical concepts should be designed in a coherent sequence ranging from the most elementary grades to the first university years.

Recent literature has evidenced a progressively increasing interest in constructs like "number sense" or "quantitative literacy," which have been promoted by the central professional organizations of mathematics education in the United States, such as the National Council of Teachers of Mathematics (NCTM), in the case of number sense, and the MAA, in the case of quantitative reasoning. These constructs have certainly evolved over time, and they try to capture the ability of students to reason qualitatively in non-algorithmic mathematical situations related to number and magnitude. It is believed that the appropriate measurement of these constructs will be useful in assessing mathematical progress of students of all educational levels. This paper capitalizes on a long-standing tradition of research in Realistic Mathematics Education and addresses completely the unanswered concerns of the San Diego conference by proposing an all-encompassing taxonomy on which to base an effective protocol for mathematics education research in number sense.

Our research interests

The research that focused our attention towards the need to develop a taxonomy for number sense originated from an effort to situate entering students to the College of Natural Sciences at the University of Puerto Rico, Río Piedras (UPRRP), in terms of their number sense skills. It was desired to correlate these skills with students' performance in the pre-calculus course (an entry level mathematics course taken by more than 90% of all undergraduates entering the college), and investigate their ability to read and understand mathematics. It should be mentioned that pre-calculus consistently shows high levels of attrition[1], in spite of the fact that these entering students are the most able and mathematically proficient in the whole University of Puerto Rico system, being second only to a few entering undergraduates admitted to some extremely competitive academic programs, such as architecture, mechanical engineering, psychology, and veterinary[2].

Mathematics professors generally regard students' inability to cope with the pre-calculus course to their lack of mathematical understanding and the accompanying absence of appropriation of the progressively complex patterns of reasoning that characterizes the acquisition of higher order thinking skills in mathematics. In studying the materials of the course, students naturally need to master a significant collection of appropriated knowledge from which they can draw and build on. Otherwise, the study of mathematics would be a never-ending collection of details. With this in mind, we developed a test with the participation of faculty members from the College of Education and the Mathematics Department of the College of Natural Sciences. The test, administered to a sample of all pre-calculus students, consisted of two parts. The first one was a survey intended to gather participants' socioeconomic information, level of education of their parents, facts relating to their home environment, whether they were first generation college students, the availability of Internet and computing resources at home (such as graphic calculators, computers), among others. The second part and main body of the test consisted of 20 multiple choice questions to determine to what extent certain knowledge, basic for the pre-calculus course, was or not appropriated by the entering students, and whether they had the resources to decide the validity of certain mathematical statements without having to resort to first principles. It is important to mention that we did not find significant differences in the performance of students from the different groups identified by the test's demographic and socioeconomic questions.

[1] Attrition refers to the percentage of students who get grades of "D" or "F", or abandon the course (officially or otherwise). For pre-calculus the average attrition during the last three years has been 65 percent.

[2] The average IGS for the different departments of the College of Natural Sciences is 324. The IGS for Architecture, Mechanical Engineering, Psychology, and Veterinary are 330, 335, 333, and 325 respectively.

We present samples of some of the reagents used and the statistics of the students' responses.

Problem 1.

If x and y are positive real numbers and $0 < y < x$, then:

a. $\frac{1}{x} > \frac{1}{y}$
b. $\frac{1}{x} < \frac{1}{y}$
c. $\frac{1}{x} < -\frac{1}{y}$
d. $\frac{1}{1+\frac{1}{x}} < \frac{1}{1+\frac{1}{y}}$

Statistics Problem 1		
a	65	41%
b	71	45%
c	9	6%
d	5	3%
N/A	9	6%
N	159	

Problem 2.

Suppose that $\frac{x}{17} = \frac{37}{y} = \frac{1}{5}$. Which of the following statements is false?

a. $\frac{37x}{17y} = \frac{1}{5}$
b. $\frac{37+x}{17+y} = \frac{1}{5}$
c. $\frac{x-37}{17-y} = \frac{1}{5}$
d. $\frac{2x+37}{y+34} = \frac{1}{5}$

Statistics Problem 2		
a	26	16%
b	17	11%
c	31	19%
d	58	36%
N/A	27	17%
N	159	

Problem 3.

If $\mu = \frac{1}{3}$, which of the following alternatives is false?

a. $\mu = 0.333 \ldots$
b. $\mu = 0.666 \ldots - 0.333 \ldots$
c. $\mu = 0.32999 \ldots$
d. $\mu = 33\frac{1}{3}\%$

Statistics Problem 2		
a	25	16%
b	21	13%
c	55	35%
d	35	22%
N/A	23	14%
N	159	

The correct answers (b, a, and c) were selected by 45%, 16% and 35% of the students, respectively. In the first problem the intention was to assess the idea of variational change of expressions as a function of the variation of their constituent variables. Trigueros and Jacobs (2008) indicate that "high school and beginning college students can work with correspondence between numbers, but the idea of variation is not easy for them" and report studies that show the difficulties of students to determine variation intervals or to think in dynamic ways, elements that are fundamental for the understanding of functions (p. 6). We feel that if students are not able to detect how certain changes in the denominator of a fraction affects its value, they will have difficulties understanding more complex variational relations.

The second problem assesses, in an algebraic setting, elements of proportional thinking as an abstraction of properties studied in proportion tables. It is important to indicate that this item should be seen as an outcome stemming from the process of verticalization in the learning of mathematics in connection, for instance, to the discussion of proportion tables.

The third problem assesses students' understanding on fraction, percentage and decimal formalism in expressing rational numbers. We consider that students entering university mathematics courses should have a fair understanding of these topics. Students' low levels of performance worried the authors of this paper.

Reflections and lessons learned

Eventually our research was diversified so as to include studies about the development of number sense in other populations, notably high school students and preservice teachers. As we examined the results we were not satisfied with the theoretical framework presented in the literature to describe and predict number sense. We embarked in a detailed analysis of possible criteria to guide research and design tests to measure number sense.

Indeed, a recurring criticism for research related to number sense pertains the way it links the topic to some structural features of our Hindu-Arabic decimal number system. But, we considered that it must include other aspects such as graphs and functions, since the latter were invented to offer numerical relations in an efficient and succinct way. There is number sense, certainly, in using graphs for discussing how much advantage should a Cheetah give to a Thompson gazelle in order to still be able to overtake it, provided we have an idea of the graphs presenting the distance traversed by each animal as a function of time (Figure 1).

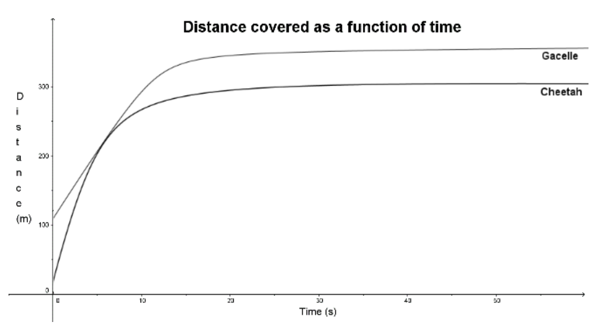

Figure 1

The graph suggests a qualitative argument that is, at best, approximate, but yet a very powerful and realistic one since the data can be (and has been) gathered by means of portable radars and recorded film data. From the graphs it can be inferred that an initial acceleration (an impressive one for the Cheetah), a peak velocity for both animals (again, very impressive for the Cheetah, the fastest animal on Earth) and, finally, a position of rest or rather a very slow movement, after the animals have used a significant part of their energy reserves and their pulmonary capacity for replenishing muscles with oxygen, as they cannot cope with the expenditure of energy.

Similarly, there are plenty of objectively identifiable elements of number sense as we look at the following climatology graphs of temperature and precipitation of two cities, A and S, and deduce, by means of the graphs, which of them is more likely to contain penguins (Figure 2).

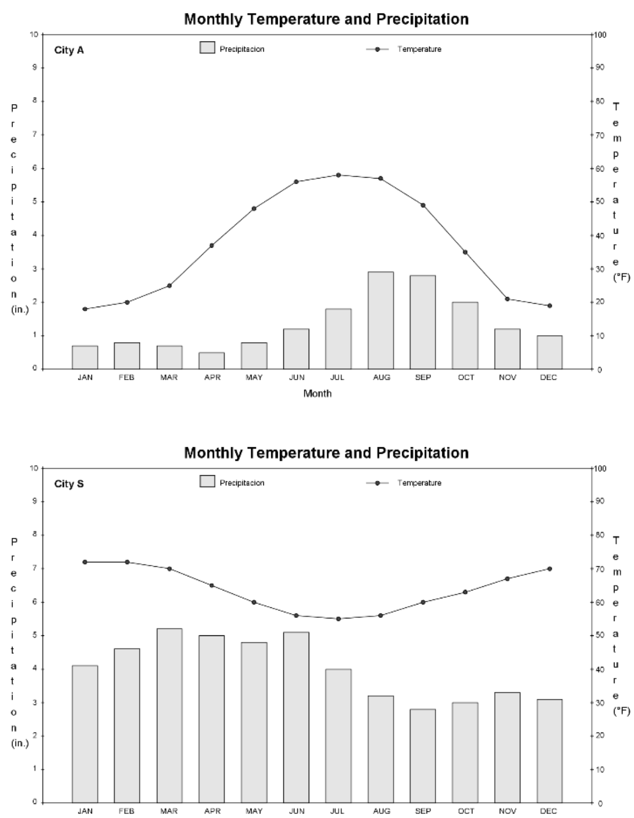

Figure 2

As a final example of the role in number sense arguments in functional contexts, let's just think of the situation in which a big building whose offices provide services that are pretty much independent of age, race or creed, is vacated, and all women in the building are gathered outside. We certainly would be very surprised if all of the vacated persons were to be of a height that is, say, between four feet six inches to five feet. That is just inconsistent with our expectation of the distribution of heights of the people in the building! We would expect a distribution more like a bell curve like the one depicted in Figure 3.

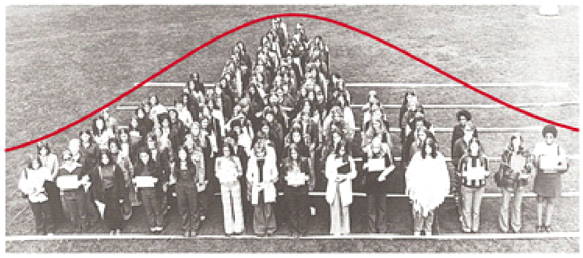

Figure 3. Courtesy *Asking About Life*, by Jennie Dusheck (reproduced with permission).

This is a consequence of the Central Limit Theorem, which happens to be a very deep and fundamental principle of mathematics. Hence, this is another instance of functional number sense. It is also an example that documents the need for a broader and more encompassing theoretical framework to guide research in this topic.

As we worked with number sense research, we have also been examining Hans Freudenthal's theory of Realistic Mathematics Education (RME). In his theory, we recognized the possibility of developing a comprehensive mathematical model for this subject. Our proposal for a theoretical framework for research in this area is based on the ideas and models of RME, and an elaboration of the concept of number sense. The notions that support our approach were originally due to Freudenthal's genius through the concept of the phenomenology of mathematical structures and his principle of guided reinvention. In the following section we summarize this theory, and then explain how it helps to develop an all-encompassing mathematical model for number sense.

Proposal for a theoretical framework for number sense research

Realistic Mathematics Education (RME) assumes that the study of the history of mathematics reveals the ways the human mind comes to organize, understand and harmonize mathematical ideas. In the words of Freudenthal (1983),

> Our mathematical concepts, structures, ideas have been invented as tools to organize the phenomena of the physical, social and mental world. Phenomenology of a mathematical concept, structure or idea means describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind…" (p. ix)

According to him, phenomenologies can be *historical,* when the description of their mathematical structures is accomplished subordinated to their historical development; *didactical,* as far as they describe the learning process of the young generation, and constitute a way to show teachers "the places where the learner might step into the learning process of mankind"; and *genetic,* if the description is with regard to the cognitive processes of mental growth (Freudenthal, 1983, p. 10). As can be expected, the history of mathematics plays a central role in the didactical phenomenology of mathematical structures to the extent that they serve the important purpose of providing an initial proposal for teaching the different areas of mathematics. Freudenthal himself worked out the details of the phenomenology of many mathematical structures, such as fractions, ratio and proportionality, geometrical structures, and several others (Freudenthal, 1983). He also discussed contexts appropriate for the development of the corresponding didactical phenomenology of these areas.

The didactical phenomenology of a given mathematical structure comes coupled with several didactical principles, the most notable one being Freudenthal's *guided reinvention principle*. The principle states, succinctly, that learning mathematics is no more and no less than reinventing mathematics under the guidance of a teacher. But he explains that what is to be learned is actually much more specific than just "mathematics", in his words: "*…the learner should reinvent mathematizing rather than mathematics; abstracting rather than abstractions; schematizing rather than schemes; formalizing rather than formulas; algorithmizing rather than algorithms; verbalizing rather than language…*" (Freudenthal, 1991, p. 49). The didactical phenomenology of mathematical structures gets coupled with the reinvention principle in the instructional design theory known as RME.

Cobb (2000) states that in RME: 1) the instructional source materials should be 'experientially real' and taken from the student's immediate context (hence the name "realistic mathematics education"); 2) starting points of the instructional source materials should be justifiable in terms of the potential ending points (in subordination to the phenomenology associated to the corresponding mathematical structures); and 3) the instructional source materials should contain activities in which students, guided by their teachers, create and elaborate symbolic models of their informal activity (in compliance with the guided reinvention principle). Furthermore, the models of contextual situations expected to be designed by the students are sometimes called "descriptive models", and the codification or organization of its mathematical content is referred to as "horizontal mathematization".

According to Van den Heuvel-Panhuizen (2000), it was Streefland who, in 1985, detected the shift in models as a crucial mechanism in the growth of understanding. Finally, the descriptive models the student constructed (with the teacher's assistance), abstracted and voided of some contextual details, gained an existence of their own and turned into "prospective models". This is the process of "vertical mathematization" discussed by Freudenthal in several of his writings (for example, Freudenthal, 1991, p. 30) accounting for the elevation of abstraction levels in learning mathematics (Gravemeijer, 1994a;

Streefland, 1991; Treffers, 1991a). The interplay of the models and the process of verticalization is discussed in Gravemeijer (1994a, 1994b).

The verticalization phase of Freudenthal's model of learning encompasses an element of the idea of number sense, and that is the student's appropriation of the elements of content and structure that allows for "seeing the forest and not only the trees". Students who succeed in the study of mathematics must be able to develop a certain "numerical sense" that allows them to identify component arguments in proofs and refutations, and to read and write mathematics efficiently.

Students develop some sort of "numeric sense" that helps them understand and integrate coherently mathematical knowledge (Marshall, 1989, 40). This "number sense" makes them more efficient in reading and writing mathematical arguments, and, most importantly, allows them to reason qualitatively in the appropriate contexts, obviating long chains of detailed argumentation that become, in some sense, automatic, that is: become part of their mathematical "common sense".

This is, really, a particular instance of a more general principle described by Freudenthal (1991):

> Through reflecting on his own activity man discovers paradigms, which are abstracted into patterns of mental action, and made conscious as schemes by which thought is organized on behalf of new progress — daptable schemes, that is, which allow for varieties of use, as well as, in the same right, rigid single-purpose schemes which, thanks to their rigidity, can lead a life of their own, called algorithms. (p. 10)

Somehow, as students, we must be able to increasingly see the forest and not just the branches of the mathematical tree. Students must move from the stages in which they read mathematics and study it with meticulous attention to detail, to the point of being able to integrate their knowledge into more encompassing bodies of connections and interrelations. As teachers, we are painfully aware that students who fail to develop the skills needed to subsume, compress and recognize some chains of reasoning that abound in the mathematics literature put themselves at a great disadvantage in keeping up with the expected pace and escalate the higher levels of mathematical knowledge expected of them. Certainly, for expediency only, if nothing else, students are expected to develop this skill in order to study efficiently and in a timely fashion, responsive to the pace of classroom discussions.

An example may be illustrative here. In an introductory analysis textbook, it is stated that if a sequence converges to a non-zero real number, then, for sufficiently large values of the sequence index, it must happen that the corresponding terms of the sequence are all distinct from zero; in symbols, if the sequence $(S_n)_n$ of real numbers converges to the real number $s \neq 0$, then for some index n_0 it must happen that $s_n \neq 0$ holds for all $n \geq n_0$. The argument given in the textbook presents a series of estimations with attention, of course, to the indices involved. The argumentation, first, states that for sufficiently large[3] values of n, we must have $\left||s_n| - |s|\right| \leq |s_n - s| < \frac{|s|}{2}$, and this is just a standard property of absolute values combined with the definition of convergence and capitalizing on the fact that $|s|$ is a positive quantity, according to hypotheses. In particular, this states that for sufficiently large values of n, $\left||s_n| - |s|\right| < \frac{|s|}{2}$, that is to say, for all sufficiently large values of n, $\frac{|s|}{2} = |s| - \frac{|s|}{2} < |s_n| < |s| + \frac{|s|}{2} = \frac{3}{2}|s|$, and, in particular, $|s_n| > \frac{|s|}{2} > 0$ for such values of n.

This is an example of an argument that presents significant difficulties to neophytes in the study of analysis, but which is very simple for students who have developed a certain degree of number sense related to the metric structure of the real number line, that is, the model of the real numbers which is embodied in the usual and ever-present number line. In fact, indices apart, all that is being said is that, given two real numbers y and c, with c positive, if the distance from y to c is smaller than $\frac{c}{2}$ then y must lie in the interval $\left(c - \frac{c}{2}, c + \frac{c}{2}\right) = \left(\frac{c}{2}, \frac{3c}{2}\right)$; see Figure 4.

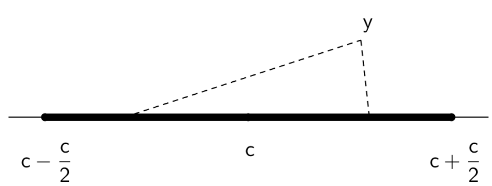

Figure 4

Of course, stated this way, it is evident that $y > c - \frac{c}{2} = \frac{c}{2}$. Taking $y = |s_n|$ and $c = |s|$ the result is obvious after some qualification regarding the index n. Of course, the student who resorts to the geometry of the number line in order to understand the argument, not only shows an attainment of a higher level of number sense than the student who must work out the tedious details of the inequalities presented in the textbook. In general, the number line is an invaluable tool in thinking about inequalities on the real line as well as the algebraic properties of the real numbers.

A taxonomy for number sense

Several researchers have remarked on the different ways to interpret the idea of "number" (Freudenthal, 1983; Kieren, 1980; Tall, 1991). Students are able to reconstruct the

[3] That is, for a "tail end" of the index set, that is, for a set of the form $\{n | n \geq n_0\}$, where n_0 is a suitably chosen index. The set of indices of a sequence is, in general, a (so called) terminal segment of integers (that is, a set of integers consisting of a first one and all others greater than it) but typically, this set is chosen to be the set of non-negative integers.

meanings associated with the different facets of number with the guidance of their teachers, who make use of models to aid in accomplishing the task. RME has been especially effective in the production of didactical models to aid the construction and reconstruction of the notion of number.

Our proposal for number sense research (relating to themes, methodologies and theoretical framework) is intimately related to RME and capitalizes on its practical implementation. We propose a taxonomy of all models of RME to identify, within each of them, the elements of verticalization and automatization that get to be incorporated into the mental structures we would identify as part of the "number sense" developed by students. We hasten to add that the "number sense" we have mentioned several times in this writing, and also gets mentioned repeatedly in other places, like Sowder and Schappelle (1989), does not necessarily refer to an innate or given ability to reason numerically. Granted, some individuals can have special numerical skills not linked, perhaps, to formal mathematical studies, but the goal of number sense research should be, in our view, that of seeking the ways in which the teaching of mathematics helps develop and automatize this set of skills in all students. It should be emphasized that we use the term "automatization" to refer to the process of appropriation of knowledge, that is, the incorporation of new knowledge to the student's cognitive resources, adding thus more fluidity and ease to the mental processes associated with mathematics learning.

The quote of Srinivasa Ramanujan's interview at the beginning of this writing illustrates the point. Ramanujan, a mathematician whose opus required in excess of a century to come to grips with it, was known for his powerful computational. For Ramanujan and other mathematicians like Archimedes, Euler, or Newton, mathematical computation and estimation came more naturally than it came to others; but far from being a part of some native number sense, it seems to have been nurtured and significantly improved with study and practice. An important tenet in our proposal is that it's possible to identify, in the multiplicity of models used in RME, those elements whose automatization through the process of vertical mathematization eventually come to constitute the elements of number sense. So, our program for number sense is clear: it behooves us to identify the elements of number sense in each of the RME models, and define the corresponding methodologies amenable for research pursuant on their study. Furthermore, the proposed taxonomy has some peculiar advantages inasmuch as it provides an alternative description of mathematics education in terms of the skills expected of students by grade and number sense attainment. This fact offers the possibility of thinking of the curriculum and expectations by grade in terms of the number sense goals of mathematics education. Certainly, an added benefit for teachers in adopting the suggested point of view is that it provides an alternative way to conceive their mission goals as teachers in terms of the development of number sense, thus guiding teachers away from those sometimes sterile and useless definitions of curriculum as themes and subthemes for discussion.

Our proposal for a taxonomic classification of the models of the didactics of mathematics follows the tenets of RME and is consistent with Freudenthal's ideas regarding the role of the history of mathematics in proposing didactical phenomenology for the study of mathematics. We propose that in scrutinizing the descriptive and prospective models of RME that lead (through the processes of horizontal and vertical mathematization) to the central models of mathematics, we find a plethora of questions pursuant to explanations of the verticalization that occurs and the implied automatization of the number and quantitative facts, patterns and relations that constitute what we call "number sense". In what follows we present a taxonomy that will serve as a guide to cover, hopefully, all of the known dimensions of number sense. We present the all-encompassing mathematical model together with the descriptive and prospective models gotten through student contextual modeling and verticalization.

T1. Taxonomic entry: The algebraic dimension of number sense

There are many descriptive models associated to didactic sequences that progressively lead to the real number line. The most primitive ones are, perhaps, the two-color differentiated bead string (in groups of five or ten) and the "empty" number line (Treffers, 1991a, pp. 40-42; Gravemeijer, 1994a, p. 120). Both come, in fact, from the counting sequence that children bring to school from home and that can end in different numbers for different students.

- The bead line is interesting because it makes students count actual beads, and there is a certain intrinsic ambiguity in this sort of counting, since numbers can denote both spaces (beads) or markings on the string (points). It is a precursor of the empty number line, and this one is a primitive version for the number line used to represent faithfully the ordering of numbers and the assignment of a unique number to each point of that line (coordinate).
- The empty number line is devoid of any notion of "distance" and thus, is very far from being a "ruler." On the other hand, as a model, it has rich algebraic possibilities in as much as numbers can be viewed as operators. The empty number line is a descriptive model useful in implementing special counting strategies, often represented by "jumps" of one, five or ten. Van den Heuvel-Panhuizen (2000, pp. 5-7) discusses what she calls the "level principle," and shows evolutionary stages from the color differentiated bead string to empty number lines.

These two models lead to other prospective ones, like double lines and tables of proportions, which these are essential ingredients, as we shall see, for configuring the "metric number" line.

There are other descriptive and prospective models associated with instructional sequences and amenable to experimentation activities to reach the automatization of data and processes that comprise what we call "number sense." The bus context models for addition and subtraction of whole numbers is an interesting descriptive model (Van den Heuvel-Panhuizen, 2000, p. 26; Gravemeijer, 1994a, p. 35). Also, pairing models to determine cardinality (numerosity) without regard to the result of a "final counting" strategy is also important in determining what is more and what is less (Gravemeijer, 1994a, pp. 27-33). Other prospective models related to the transition from "number" to "algebra" are related to the study of number patterns, graphs and variational thinking (Kindt, Dekker, & Burrill, 2006).

T2. Taxonomic entry: The metric dimension of number sense

As indicated in T1 the real number has been seen as evolving from the string of beads "line" and the empty number line. It has an important mathematical feature: every point in the line corresponds to a unique real number (its coordinate), and all real numbers are coordinates of points on the line. Also, the order comparison between coordinates of points is consistent with the relative positions of the corresponding points, those lying leftmost being the smaller. However, a second property gives this number line its right to be called "metric" and that is that the distance between two points corresponds to the difference of their coordinates. Of course, the model of the metric line, as opposed to the empty number line, incorporates important algebraic and proportional thinking elements (Van Galen et al, 2008, pp. 29-33; Van den Heuvel-Panhuizen, 2000).

- It's clear that a lot of what we refer to as number sense is intimately linked with our Hindu-Arabic number system, based on the amplification or contraction of the number line unit by factors of 10 (amplifications) or subdivisions obtained by dividing by 10. Empty portions of the metric number line can do a lot for the understanding the real line decimal structure. For instance, in Figure 5 values for the real numbers A and B are given, and the student is expected to find the number corresponding to C. For instance, if $A = 2$ and $B = 6$, then $C = 9$, or if $A = 0.1$ and $B = 0.3$ then $C = 0.45$. The amplification/division models of the metric number lines are examples of density models and they can contribute significantly to the understanding of the rational number system and the nature of irrationalities on the real line.

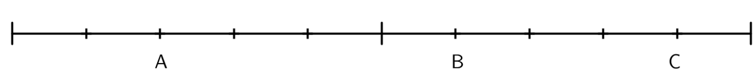

Figure 5

- Also, the geometry of the real line accounts for a significant source of "proofs" and "refutations" for many properties of the real numbers. For instance, the decimal representation of a real number x on the unit interval of the real line (for convenience we suppose that x is not a decimal fraction) says something very specific about its placement on the infinitely divisible unit interval of the number line. For example, if the first decimal digit of x is 2, then the number is located in the third interval of the subdivision of the unit interval in 10 equal parts. If we proceed with the division of this latter interval in 10 homogeneous subintervals and the number were to be located, say, on the *sixth* interval, then the second digit in the decimal expansion must be a 5 since we must have

$$x \in \left[\frac{2}{10} + \frac{5}{10^2}, \frac{2}{10} + \frac{6}{10^2}\right), \text{ and so on.}$$

There is an obvious exception to keep in mind, and that is that the decimal fractions have two decimal expressions, one of them infinite[4]; for instance,

$$0.999 \ldots = 1 \qquad 0.25 = 0.24999 \ldots$$

- Among the descriptive models of mathematics education that get into the eventual development of the metric number line are, as mentioned before, ratio or proportion tables, and calibrated double number lines. In the treatment of fractions, the context of fair division is employed and the above-mentioned models are used to establish the corresponding relationships between fractions, percentages and decimals. Fair division problems were solved very early in the history of mathematics as it is recorded in the Rhind papyrus. From these ideas we nurture the concept of infinite divisibility and tie it up with our decimal notation to get a very accurate idea of the nature of rational and irrational numbers. Some of the relevant descriptive models are discussed by diverse authors (Van Galen et al., 2008; Van den Heuvel-Panhuizen, 2000; Van den Heuvel-Panhuizen, 2003). It should be mentioned that Freudenthal (1973, pp. 197-210) and Freudenthal (1999, pp. 1-33) work out the didactical phenomenology of magnitudes in general and length in particular, and all the models mentioned before are consistent with that phenomenology. That is to say, they capitalize on models that, through the reinvention principle, emulate the mental operations associated with measurement, which, succinctly put, are the operations of rational numbers on magnitudes (amplification and subdivision of intervals of measurement) and the change of gauge (double lines).
- Floating Point Arithmetic (FPA) used in computing instruments like calculators and computers, can be useful descriptive models for understanding the metric number line. For instance, in FPA there are

[4] This seems obvious and yet, some graduate students in mathematics find great difficulties understanding the structure of Cantor sets and their representation as base three decimals (ternary expansions) when the very construction of the Cantor ternary set depends on the infinite divisibility by three of the unit interval.

intervals whose midpoints coincide with one of the extremes of the interval, and in FPA there is a *nonzero* digital number μ with the property that $1 + \mu = 1$. This weird fact is related to the "machine error", and every computer implementing a FPA system has one. It indicates the computing limit of the machine. Students can understand real numbers better if they use the calculator as a descriptive model for the metric number line.

- Also, in Dutch RME textbooks teachers can find concept-integration exercises like the following one:

Given that $\frac{324}{4} = 81$, fill in the second column of the following table:

2.5×324	$=$
25% of 36	$=$
3.24×0.25	$=$
$3.24 \div 4$	$=$
324×0.0025	$=$
36% of 25	$=$
$\frac{1}{4} \times 8100$	$=$
25×36	$=$
0.0025×0.0324	$=$

This type of exercise is very useful to apply the properties of the arithmetic operations, and to get the desired automatization of procedures relating notational representations of fractions, decimal and percentages.

T3. Taxonomic entry: The variational dimension of number sense

Tables give rise to the idea of relations between variables, and the relation between variables give rise to functional relationships and graphs. This is one of the more interesting parts of number sense, and there are a lot of descriptive models in the literature. Notable in this respect is, again, the work of the researchers at Freudenthal Institute as recorded (Kindt, 2004, 2009; Kindt, et al., 2006). From their work, a reasonable working hypothesis that comes up is that students go from patterns to tables, and from tables to sequences and functions; also, that inductive sequences are important in mastering the notion of function (Kindt, 2004, pp. 23-37, 50-54, and 64-65). Descriptive models in the given references include arrow diagrams, algebraic operations flow charts, and function machines. Algebra relationships are promoted by applying number line models, as well as the relation between plane geometry and algebra. In this taxonomic entry we also include models of continuous quantities that allow for qualitative numerical reasoning having to do with the existence of certain numbers, for instance, the existence of a real number whose square is two, and the existence of two antipodal points on a single meridian of the Earth's surface which have the same temperatures.

- Variational number sense includes automatic recognition of the fact that for a positive number a with $a < 1$, we have $a^2 < a$; or that for positive numbers x we have,

$$\frac{1}{1+x} < \frac{1}{x}.$$

Visual examinations of some algebraic expressions should automatically yield for the student information about its growth properties for some values of the variables.

- Descriptive models for data can include verbal descriptions, tables and graphs. Students first make graphs to capture the behavior of variables; conversely, by examining specific graphs, they should be able to infer functional relationships between the variables. Discussions of continuous processes (in reality, applications of the intermediate value theorem for continuous functions) are useful as descriptive models for the solution of problems about the existence of irrationalities. For instance, in Figure 6 the drafting square moves along the line $y = x$ in the first quadrant, getting away from the origin.

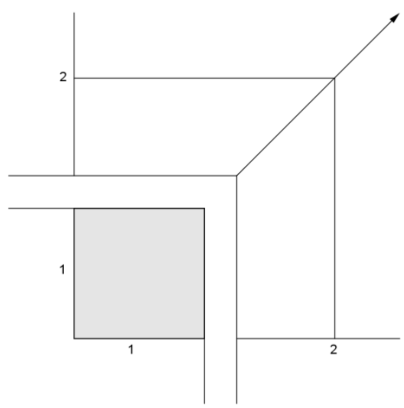

Figure 6

Since the movement is continuous and describes a square of unit area and another of area 4, it must also describe a square of area 2 whose side is $\sqrt{2}$.

T4. Taxonomic entry: Number sense associated to probability and statistics

Statistics in the lowest grades should include exclusively the study of tables and graphs, and the representation of data. It should be remembered that the birth certificate of statistics consists of tables of mortality published by John Gaunt in the 17th century to impress on the king the possibility of regulating work and production in terms of how long people had productive lives. Probability was initially interpreted as observed frequencies of certain events and got turned into abstract mathematics by employing discrete counting models in some cases, and distributions and densities in others (probabilities that are discrete or have a "continuous part"). Some aspects of the historical and didactical phenomenologies of probability and statistics are presented in Freudenthal (1973). He remarks that with the availability of functions, it is possible to make significant strides in the mathematization of probability. Also, there are obvious elements of numerical sense related to the nature of data whose measures of central tendency we know because they are given, or by the application of results like the law of large numbers or the central limit theorem. In the latter case we offered an example related to the normal distribution as a limiting situation of bernoullian probability distributions.

- Discrete probability starts with prospective models to count efficiently. This progresses into situations in which geometrical probability is used to calculate frequencies. Finally, the general situation gets described by distribution and density models.
- Estimation and probability are similar to one another with respect to number sense, although their mental actions to get probability estimates and estimations are quite different. Nevertheless, both categories of number sense address the mathematical question where a determinate answer is impossible or, plainly, unknown.

T5. Taxonomic entry: Number sense associated to estimation

Estimation is seldom a central topic of discussion in the RME literature. Our tenet is that it uses the same models as the other taxonomic entries for number sense, except that in the context of estimation, determinate models are used but without exact information, that is, with approximate information. The only exception to this is, of course, probability, which provides estimated answers to contextual situations that do not admit definite answers. Granted, this presumably imposes particular research methodologies for determining the verticalization attained as such descriptive models turn into prospective ones. In this situation, the mathematization of the process of estimation has to be sorted out from the vertical mathematization that stems from the original deterministic model. An example is presented to illustrate the point.

- The following is an illustration on a first grade book that intends to engage students in a conversation about how to count objects in a toy shop (Figure 7). The diagram shows two kids looking at some displays in the store. Students are asked, in one case for the number of teddy bears, and in the other for the number of toy cars.

Figure 7. Courtesy *Asking About Life* (2^{nd}. ed., p. 394), by Allan Tobin & Jennie Dusheck.

For the purposes of counting, we can ask the indicated questions and expect an answer under the assumption that what the students cannot see follows the same pattern as what in fact they can see. But if we don't assume this, then the problem does not have a uniquely determined answer, and thus it is an estimation problem. In counting, the students argued for a pattern that applies to what they don't see, while in the estimation problem they have to think of lower and upper bounds for the objects they cannot see.

Conclusion

Originally, our interest for embarking on our research on number sense came about as an attempt to better comprehend the difficulties students experience in the construction of the fundamental notions needed to understand functions and calculus. Our observations evidenced the fact that they had shortcomings in grasping algebraic order of the real numbers, the density of the rational numbers, and the properties of the usual operations in the different numerical systems. Our first approach to this problem was to try to measure the number sense skills entering students to the university brought with them from high school. Upon revising the pertinent literature, we found that the construct of number sense was introduced in the 1980s, and even today the notion can still be better defined. Similarly, we found that the corresponding research questions and the associated theoretical frameworks were not well established or agreed on. In 1989 the National Science Foundation sponsored the Conference on the Foundations for Research on Number Sense and Related Topics mentioned before and, as we have seen, the results did not improve significantly the state of the research on this subject. As these difficulties and subsequent developments confirm, mathematics education research has been unable to give satisfactory answers to questions related to what is number sense, how do we measure and assess it, how do we teach it at school, how it is linked to mental computation and estimation, and what research frameworks could be regarded as more convenient to carry out research on this matter.

In this work, we propose RME as the appropriate theoretical framework for number sense research, coupled with a taxonomy of all descriptive and prospective models of RME, with the aim to identify within each of them the elements of verticalization and automatization that get to be incorporated into mental structures that we would identify as part of what we call number sense. This approach can give a view of the curricular contents of mathematics education in terms of the expectations by grade of number sense skills attainment. This will give teachers a more realistic and productive view of the curriculum.

RME underscores need to revisit and reflect on the topics in order to accomplish vertical mathematization, reaching higher levels of understanding in specific mathematical areas. Treffers (1991b, p. 24) refers to this point precisely, and remarks that verticalization occurs over long periods of time and revisiting mathematical themes along the different stages of education. Apropos of the role of models, this implies that the models of mathematics education get progressively verticalized in the process of developing higher order levels of understanding and abstraction. Implicit in this verticalization, there are automatization elements on features of structural model, as well as of mental processes. Such automatization empowers students to add to their knowledge efficiently as they are able to muster the automatized resources and use them as basic or given elements in the subsequent use of models to *describe* new mathematical situations. It behooves researchers to describe, in more precise terms, the process of verticalization in the didactics of mathematics and how this process adds to the appropriation and automatization of mathematical knowledge. Our taxonomy could result useful in the realization of this work.

The data collected is limited to students of the pre-calculus course at UPR-Rio Piedras Campus; this can be considered a limitation of this study. However, the theoretical foundation allows us to ensure that the proposed taxonomy is applicable to all educational levels. Even more, we consider imperative the inclusion of the issues of numerical sense, as proposed, in the intermediate and higher-level school curricula.

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