Diálogos **79** (2002), pp. 319–371.

SECOND-ORDER ABSTRACTION, LOGICISM AND JULIUS CAESAR (I) [1]

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Ein Wort ohne bestimmte Bedeutung hat für die Mathematik keine Bedeutung.

(Frege, 'Über die Grundlagen der Geometrie" I, (1906), KS, p. 290)

Second-order abstraction was Frege's device *par excellence* for introducing the objects of arithmetic. He believed that the numbers had to be defined as purely logical objects in order to establish the thesis that arithmetic is a branch of logic. In *Die Grundlagen der Arithmetik*, Frege's attempt to introduce cardinal numbers by means of what has come to be known as "Hume's Principle" foundered, by his own lights, on the pervasive referential indeterminacy of numerical terms to which that principle gives rise. This problem is generally called "the Julius Caesar problem". Frege intended to solve it by making a transsortal identification, namely by defining the number of Fs as an equivalence class of equinumerosity. However, this explicit definition rested on the questionable assumption that the reader knows what the extension of a concept is. In *Grundgesetze der Arithmetik*, Frege was fully aware that

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[1] I would like to express my gratitude to Roberto Torretti for his generous help and useful advice over nearly twenty years. — I discussed an earlier and shorter version of this paper at an international conference on language, logic and artificial intelligence in Fortaleza (Brazil), the Catholic University of São Paulo, the National University of Costa Rica (Department of Mathematics), the universities of Innsbruck and Zagreb and at the Philosophical Faculty in Zadar (Croatia). Warm thanks especially to Goran Svob, Slavko Brkic, Reinhard Kleinknecht, Mario González, Lafayette de Moraes, Max Freund, Daniel Vanderveken, Hans Kamp and John Searle. I am also grateful to Álvaro López Fernández for his patience and encouragement, while I was trying to finish this paper for publication.

the viability of his logicism depended crucially on the introduction of extensions in a methodologically sound and purely logical fashion: all numbers were to be defined as extensions of concepts or, in more general terms, as courses-of-values of functions. Yet in introducing the courses-of-values via Axiom V — the exact structural analogue of Hume's Principle — he encountered a formal version of his old Caesar problem from *Grundlagen*.

This is the first part of an essay consisting of three parts. In this part, I want to bring into focus and assess critically the Caesar problem in *Grundlagen* in the light of Frege's logicist enterprise. In particular, I argue (a) that the Caesar problem, which is supposed to stem from his tentative inductive definition of the natural numbers, is only spurious, not genuine; (b) that the genuine Caesar problem deriving from Frege's attempted contextual definition of the cardinality operator is a purely semantic one; (c) that the explicit definition of the cardinality operator is intended not only as a means of removing the Caesar problem, but also as a means of saving the analyticity of Hume's Principle as the pivot of the formal derivations of fundamental theorems of cardinal arithmetic; (d) that Frege's envisaged contextual definitions of fractions, irrational numbers and complex numbers using second- or higher-order abstraction would lead to a whole family of Caesar problems all of which are supposed to be resolved by setting up appropriate explicit definitions of these numbers; (e) that the prospects of overcoming the Caesar problem by explicitly defining cardinal numbers as objects which are not classes appear to be poor, contrary to what a puzzling remark by Frege at the end of *Grundlagen* seems to suggest. In the first two sections of this paper, I try to shed light on the nature of Fregean abstraction and Frege's notion of logical object.

1. Setting the stage: Fregean abstraction principles

Frege uses the word "abstraction" for the most part in a psychological sense, especially when severely taking to task certain rival theories of arithmetic. In contrast to psychological abstraction, which he regarded as a thorn in the flesh and combatted vigorously, his own contextual method of introducing abstract objects or of bestowing a reference upon abstract singular terms by appeal to an abstraction principle may simply be called *Fregean abstraction*. A schema for a Fregean abstraction principle can be stated as follows:

(S) $Q(\alpha) = Q(\beta) \leftrightarrow R_{eq}(\alpha, \beta)$.

Here "Q" is a singular term-forming operator, α and β are free variables of the appropriate type, ranging over the members of a given domain, and "R_{eq}" is the sign for an equivalence relation holding between the values of α and β. In *Die Grundlagen der Arithmetik* (GLA), the paradigms for Fregean abstraction are:

(1) $D(a) = D(b) \leftrightarrow a \parallel b$.

The direction of line a is identical with the direction of line b if and only if line a is parallel to line b.

(2) $N_xF(x) = N_xG(x) \leftrightarrow E_x(F(x),G(x))$.

The number of Fs is identical with the number Gs just in case F and G are equinumerous (i.e. according to Frege's definition of equinumerosity in §72 of *Grundlagen*: if and only if there is a relation R which correlates one-one the Fs and the Gs).

In *Grundgesetze der Arithmetik* (GGA), the paradigm is

(3) $\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha) \leftrightarrow \forall x(f(x) \leftrightarrow g(x))$.[2]

The course-of-values of the function f is identical with the course-of-values of the function g if and only if f and g are coextensional (or coextensive).

(3) is the famous-infamous Axiom V of Frege's logical system in *Grundgesetze*, the structural analogue of (2). George Boolos has baptized (2) "Hume's Principle"; I shall follow him in using this name, because it has become familiar in the Frege literature and does not, in my opinion, give rise to confusion[3]; I shall henceforth use "HP" as an abbreviation for "Hume's Principle", however. In (1), the direction operator "D(x)" operates on singular terms, in (2) the cardinality operator "$N_x\varphi(x)$" operates on one-place predicates of first level, and in (3) the course-of-values operator "$\dot{\varepsilon}\varphi(\varepsilon)$" acts on monadic function-

[2] The right-hand side of this equivalence is rendered in modern notation here. Instead of the sign "↔" Frege uses "=". In his theory of sense and reference after 1891, Frege regards expressions, which have the syntactic structure of sentences and are referential, as proper names of the True or the False. He uses, of course, a different symbol for the universal quantifier.

[3] See, however, Michael Dummett's protest against the use of the name "Hume's Principle" in Dummett (1998), pp. 386f.

names of first level. (1) is a first-order abstraction principle (parallelism between lines is a first-level relation), while (2) and (3) are second-order abstraction principles (both equivalence relations involved are of second level).[4] In contrast to the first-order principles, the principles of second- or higher-order involve a "projection" from the larger domain of concepts (or functions) into the smaller domain of abstract objects of a certain kind; and the latter may, of course, fall under the former (if they are of first level). It is this feature of the higher-order principles that makes them fairly powerful, but at the same time susceptible to logical difficulties. While (1) does not require the existence of any more of abstracta (directions) than there are lines, (2) requires the existence of n + 1 abstracta (cardinal numbers), given n objects of the original kind. In fact, the introduction of the cardinality operator relies crucially on the assumption that the domain of the first-order variables is infinite. As far as (3) is concerned, Frege had to learn that the demand it makes on the size of the domain is not realizable. If n objects are in the domain, (3) requires the existence of 2^n abstracta. If the number of abstracta introduced via an abstraction principle exceeds the number of objects in the domain, as is the case with (3), we may term the principle *inflationary*. It is a necessary condition for the truth of an abstraction principle that it be non-inflationary.[5] The theory which results from adjoining HP to standard axiomatic second-order logic — also referred

[4] Note that (2) is a schematic formulation of HP; in this formulation, both sides of HP are (closed) sentences. Here "F" and "G" are schematic letters for monadic first-level predicates, not variables for first-level concepts. By way of contrast, in "$\forall F \forall G(N_x F(x) = N_x G(x) \leftrightarrow E_x(F(x),G(x))$", "F" and "G" are variables for first-level concepts; here we have the universal closure of the open sentence "$N_x F(x) = N_x G(x) \leftrightarrow E_x(F(x),G(x))$". Analogous remarks apply, *mutatis mutandis*, to Axiom V. In (3), "f" and "g" are schematic letters for monadic functional expressions of first level, while in the universal formulation of Axiom V, "f" and "g" are variables for one-place functions of first level.

[5] Kit Fine (1998, p. 510) proposes, by appeal to an informal concept of truth, that an abstraction principle will be true if and only if its identity criterion is non-circular and yields a non-inflationary and predominantly logical equivalence on concepts. He calls an abstraction principle *predominantly logical* if its identity criterion involves only a "small" number of objects in relation to the number of objects in the universe as a whole. Notice that the notion of being small that Fine uses here is not the usual one. A subset C of cardinality $\mathbf{c}$ is said to be small relative to a domain D of cardinality $\mathbf{d}$ if $\mathbf{d}^{\mathbf{c}} \leq \mathbf{d}$, i.e. if the number of subsets of the same cardinality as the given subset does not surpass the cardinality of the domain itself. See Fine's comparison of two model-theoretic criteria of acceptability for abstraction principles with the aforementioned informal criterion, pp. 511ff.

to as Fregean or Frege Arithmetic (FA) — is probably consistent,[6] but second-order logic plus Axiom V is inconsistent.[7]

The right-hand branch of each biconditional (1), (2), and (3) embodies a specific criterion of identity for the abstracta of the corresponding type: parallelism, equinumerosity, and coextensionality. A Fregean abstraction principle thus satisfies what Frege considers to be the cardinal prerequisite for any methodologically sound introduction of abstract or logical objects. It is true enough, though, that in §§67-8 of *Grundlagen* he rejects (1) and (2) as contextual definitions of "D(x)" and "$N_x\varphi(x)$" respectively, precisely because he holds that in either case the criterion of identity fails to do what it ought to do, namely to cover all conceivable cases. According to Frege, (1) determines the truth-value of only those equations in which the expressions flanking "=" are both direction terms of the form "D(b)". Analogous remarks apply to (2). Earlier, at the outset of §62 of *Grundlagen*, Frege had raised the epistemological question as to how numbers are to be given to us if we do not have any cognitive access to them through ideas or intuitions. It is

[6] Cf. Boolos (1987), (1987a) and (1990). The equiconsistency of second-order arithmetic (i.e. analysis) Z_2 and FA can be proved in Primitive Recursive Arithmetic (PRA). In view of the equiconsistency of FA and Z_2 we may say that all arithmetical theorems which Frege proves in *Grundgesetze* are proved in a consistent subtheory of his formal theory (cf. Heck (1998), p. 430). In his paper 'Is Hume's Principle Analytic?' (1997), Boolos argues that we do not *know* that Z_2 is consistent and that it would appear to be a genuine possibility that the discovery of an inconsistency in Zermelo-Fraenkel set theory (ZF) might be refined into that of one in Z_2. He concludes that we are uncertain whether FA is consistent (pp. 259ff.) Compared with what Boolos wrote in 'The Consistency of Frege's *Foundations of Arithmetic*' (1987), this is a surprising change of mind, completely passed over in silence by him. On this issue see the successor to this paper.

[7] However, the first-order fragment of this theory is consistent, as first demonstrated by T. Parsons in his 1987 paper; see also Dummett (1991), p. 219. Without second-order quantification, the introduction of courses-of-values would be pointless, however, because membership would be indefinable for Frege. R. Heck (1996) shows, by way of extending Parsons's proof, that both the simple and ramified *predicative* second-order fragments of the logical system of *Grundgesetze* are consistent. Moreover, he establishes that Robinson arithmetic (Q) is relatively interpretable in the simple predicative fragment. Parsons's proof for the first-order fragment of Frege's logical theory as well as its extension carried out by Heck are model-theoretic and nonconstructive and cannot be formalized even in first-order Peano Arithmetic (PA). In a recent note, J. Burgess (1998) gives a constructive proof of Parsons's result. Concerning Heck (1996), see K.F. Wehmeier (1999). Wehmeier proves the consistency of Δ^1_1-comprehension with Frege's Basic Law V, thereby refuting a version of a conjecture of Heck (1996).

precisely at this point that his requirement of stating a sufficiently general condition for the identity of directions or cardinal numbers comes into play. Frege, as a matter of fact, presents the demand as a special case of what, he thinks, is required in general if we are to use a singular term to refer to an object, be it concrete or abstract. "If for us the symbol *a* is to denote an object, we must have a criterion which decides in every case whether *b* is the same as *a*, even if it is not always within our power to apply this criterion" (GLA, p. 73). From the restricted applicability of the criterion of identity for directions or cardinal numbers arises his so-called Julius Caesar problem[8], a version of which he also had to face in *Grundgesetze* regarding (3). On occasion, I shall refer to this problem and its variants as "Frege's indeterminacy problem".

The cardinality operator, unlike the direction operator, is a *second*-level function-name; correspondingly, equinumerosity is, in contrast to parallelism, an equivalence relation between (first-level) *concepts*. It is for this reason that the proposed contextual definition of "$N_x\varphi(x)$" through (2) (henceforth referred to as "(CD)"), unlike the proposed contextual definition of "D(x)" via (1), gives rise to a difficulty that seems to have gone unnoticed by Frege.[9] While sentences involving the notion of direction could always be translated into sentences not involving it, we cannot do the corresponding sort of thing for sentences involving the notion of number. (CD) provides no way of eliminating the cardinality operator in all cases in which it is applied to a predicate itself containing one or more occurrences of "$N_x\varphi(x)$". In general, (CD) fails to supply any means of eliminating "$N_x\varphi(x)$" from an expression of the form "$N_xF(x) = x$", where "x" is either a free or a bound variable.

1.1 Abstraction in Grundlagen

In *Grundlagen*, Frege maintains that one and the same content can be carved up in distinct ways and so emerge in sentences of different logical form:

> The judgement 'The straight line *a* is parallel to the straight line *b*', in symbols $a \parallel b$, can be construed as an equation. If we do this, we obtain the concept of direction, and say:

[8] The heading of §66 of *Grundlagen* is: The criterion of identity is insufficient.

[9] Of course, if (1) or (2) is referred to as a (tentative) definition, the sign "$\leftrightarrow$" ought to be replaced with, e.g., ":=".

> the direction of the straight line *a* is identical with the direction of the straight line *b*.' We replace the symbol || with the more general symbol =, by distributing the particular content of the former symbol to *a* and to *b*. We split up the content in a way different from the original way, and thereby obtain a new concept (*Grundlagen*, §64).[10]

In this passage, Frege describes the transition from one mode of speaking (A) to another (B), that is, the abstraction step, in somewhat vague terms. The equivalence-statement (A) on the right-hand side of "↔" is designed to state the condition under which the identity-statement (B) is true, thereby fixing the reference of a contextually introduced term, the direction operator. (B) and (A) are considered to have the same sense or content. And it seems that for Frege our grasp of the content of the new sentence, that is (B), is mediated by our recognition of its having the same content as the old one, that is, (A). Yet, contrary to what he contends, (A) cannot be construed as an identity-statement, at least not in a strict sense. Even if we accepted his stipulation that (A) and (B) are to have the same content, we could only grant that (A) can be transformed into (B).

As far as the second half of the quoted passage is concerned, I find it hard to follow. How are we to understand Frege's talk of carving up the same content in a way different from the original way, thereby obtaining a new concept? On the face of it, it might suggest that what he has in mind here is the purely syntactic device, always endorsed by him, of removing some or all occurrences of an expression from a sentence, and of marking the resulting gap(s) as an argument-place of the appropriate type, leaving a function-name (concept-expression or relation-expression).[11] This operation (call it *gap formation*) is supposed to go hand in hand with the process of analyzing or dividing a thought into thought-components (call this *decomposition*). Closer examination, however, reveals that the recarving of the content of (A) does not involve gap formation. First, the transition from (A) to (B), as described by Frege, is not a purely syntactic operation. It is rather a mixed operation, carried out both on the level of signs and on the level of content.[12] In particular, replacing a symbol with a different symbol has

[10] For the most part, I have modified the existing English translations of Frege's works.

[11] The procedure of gap formation is described in precise and more general terms in §26 of *Grundgesetze*, vol. I.

[12] What is it to mean precisely that the particular content of "||" is distributed to

nothing to do with gap formation. Second, we can neither extract from (A) the direction operator nor can we discern in (A) the equality sign.[13]

Clearly, the division of a sentence-content (or thought) into content-components (or thought-components) is inextricably intertwined with the analysis of a sentence which expresses it; and it is in terms of gap formation that the latter operation proceeds. Or is there any device distinct from decomposition which Frege could justifiably call a *dissection* of a sentence-content (or thought)? I do not think so. In short, instead of characterizing the transition from (A) to (B) as resting on a dissection of a content in a way different from the original one, Frege should really have said that one and the same content is presented in two different ways.

1.2 Abstraction in Grundgesetze

In the second volume of *Grundgesetze*, when Frege comes to consider more closely the transformation embodied in Axiom V(cf. GGA II, §§146f.), his mode of speaking has changed. He now mentions neither the syntactic operation of replacing the sign "R_{eq}", which denotes the equivalence relation on the right-hand side of "$Q(\alpha) = Q(\beta) \leftrightarrow R_{eq}(\alpha, \beta)$", with the more general sign "=" on the left-hand side, nor the semantic operation, germane to the former, of distributing the content of "R_{eq}" to α and β.[14] Likewise, he no longer says that we gain a new concept by splitting up a sentence-content or thought in a fashion different from the original one. I presume that even in *Grundlagen* Frege

a and to b? Frege hardly provides a clue for answering this question.

13 What is the original way of splitting up the content that (A) and (B) have in common supposed to be? The thought expressed by the original sentence (A) could be decomposed, for example, into the senses of "a", "b" and "||" by analyzing (A) into just these three expressions. It would be awkward, however, to see this operation as bringing it about that the thought in question emerges in (A).

14 Actually, in Frege's notation, the right-hand branch of Axiom V already contains the identity sign. In his fragment 'Ausführungen über Sinn und Bedeutung', he uses a special symbol designed to denote the relation of coextensionality of first-level functions or concepts which corresponds to, but should not be confused with, identity between objects (cf. NS, p. 132). Frege emphasizes that if instead of this special symbol we use his ordinary notation (employed on the right-hand side of Basic Law V), we have the same second-level relation; but the sign of identity does not suffice on its own to denote this relation. Rather, it has to be combined with the sign for generality, because the relation of mutual subordination or coextensionality of first-level concepts is, in the first place, a generality, not an identity. As is well known, Frege holds that concepts cannot stand in the relation of identity.

would have renounced transferring his mode of characterizing first-order abstraction in terms of the twofold operation just specified to the case of second-order abstraction. Certainly, he was well aware that the move from right to left in a second-order abstraction principle, unlike that in a first-order principle, involves stepping down from level two to level one. And I imagine that by the time he wrote *Grundgesetze* he had probably refrained altogether from characterizing abstraction along the lines of §64 of *Grundlagen*.

Be this as it may, in the second volume of *Grundgesetze*, Frege describes the transition from an equivalence relation to an identity of abstract objects briefly as follows (I mean to apply his description also to HP, structurally the nearest kin of Axiom V): In carrying out the transformation, we are recognizing something common to the two functions or concepts. Thus, the step of abstraction proceeds in such a way that we assign the same object to the coextensional functions or equinumerous concepts, namely the same course-of-values or the same cardinal number. At one point (NS, p. 198), Frege says that by converting an equivalence relation into an identity we acknowledge that there is exactly one object which the two singular terms flanking the identity-sign denote. Yet acknowledging the existence of exactly one abstractum associated with the items of the equivalence relation does not involve, in his view, that a new object has been brought into being. In *Grundgesetze*, vol. II, §147, Frege takes pains to meet this possible objection, but does so a little half-heartedly. He first dissociates himself from other mathematicians by pointing out that (3) serves towards the ends that they intend to attain by creating new numbers. He then asks: "Can our procedure be called creation? Discussion of this question may easily degenerate to a quarrel over words. In any case, our creation, if you like to call it that, is not unbounded and arbitrary; the way of performing it, and its admissibility, are established once and for all." Several other pronouncements Frege made during both the *Grundlagen* and the *Grundgesetze* periods leave no doubt, however, that he considered numbers and extensions of concepts (or courses-of-values) not to be creations of the human mind in any reasonable sense of the word "creation".[15] By performing Fregean abstraction, we merely *conceive*

[15] Cf., e.g., GLA, §§96, 105; GGA I, p. XXIV; KS, pp. 122f.; NS, pp. 87, 144f., 149, 160, 214.

these objects, *get hold of* them, and that is the end of matter.[16] In this respect, Frege differs fundamentally from Dedekind, his fellow-combatant for logicism.

1.3 Does Fregean abstraction involve something like a "reconceptualization"?

Crispin Wright (1997) and (1999) has suggested that the key idea of the intention of the neo-Fregean in laying down HP as an explanation designed to fix the concept of cardinal number is this: an instance of the left-hand side of an abstraction principle is meant to incorporate a *reconceptualization* of the type of state of affairs depicted on the right: "Numbers are, rather, like directions, the output of a distinctive kind of re-conceptualization of an epistemologically prior species of truth" (Wright (1999), p. 209). Expressing it in this rather vague way, Wright seems to rely on Frege's mode of characterizing first-order abstraction in *Grundlagen*, but I fail to see that the description of Fregean abstraction in terms of "reconceptualization" should be illuminating in any way. My reserve applies especially to the case of second- or higher-order abstraction.

Charles Parsons (1997, p. 270) seems to accept Wright's idea that Fregean abstractions effect a reconceptualization for the case of first-order principles. "In those cases," he says, "it seems that what we are doing is simply individuating the objects we have in a coarser way, one might say carving up the domain, or a part of it, a little differently." This is in no way clearer than Wright's characterization, though. Fine (1998, p. 532) introduces the term "definition by reconceptualization" and says that it rests on the idea that new senses may emerge from a reanalysis of a given sense. He further asserts that the idea derives from §§63-64 of *Grundlagen*. He makes the proviso, however, that his aim is not to be faithful to Frege's thought. The result of his analysis is that the possibility of definition by reconceptualization can probably take us no further than an implicit definition of a standard sort. As I have argued above, it is difficult to make sense of Frege's mode of characterizing first-order

16 Frege's use of the German expressions "fassen" ("to conceive" is probably the appropriate English word here) or "sich bemächtigen" (I suggest rendering this phrase through "to get hold") in this specific context is not as clear as it should be. Frege never endeavoured to explain our supposed epistemic relation to logical objects in more precise terms.

abstraction in terms of splitting up a content in a manner different from the original one. Likewise, it seems inappropriate to me to describe this operation in such a way that new senses may emerge from a reanalysis of a given sense. It is rather Frege's method of gap formation, combined with his principle that a thought can be decomposed in distinct ways, where the terms "reconceptualization", "reanalysis" or "recarving" could be applied appropriately. But again, gap formation and decomposition should not be confused with the operation Frege intends to describe in §64 of *Grundlagen*. The former operations are not even akin to the transformation from right to left in a Fregean abstraction principle.

Returning to Wright, I wish to draw attention to another controversial point in his account. It is his claim that Fregean abstraction involves the formation, actually the *creation* of a concept, if it is to be faithful Frege exegesis.[17] In a sense, we can say that the transition from right to left in the abstraction principles (1) – (3) by presenting a sentence-content in a guise different from the original one involves a kind of concept introduction. Yet, in saying this, we must bear in mind that it has nothing to do with what in *Begriffsschrift*, 'Booles rechnende Logik und die Begriffsschrift' and in *Grundgesetze* Frege regards as a kind of genuine concept *formation*.[18] I mean, of course, the syntactic process of gap formation, which may also be, termed the *method of the extraction of concept- and relation-expressions* (more generally: of functional expressions). Here we may ignore the fact that in *Grundgesetze* Frege uses more precise terminology than in *Begriffsschrift* and his essay on Boole's and his own logic. In *Grundgesetze*, he has to account for another kind of concept formation besides gap formation: a complex concept-expression (monadic function-name) of first level can be formed by inserting an object name into one of the argument-places of a relation-expression (dyadic function-name) of first level. Thus, the rules of gap formation and insertion are the only explicitly stated rules which govern concept formation (or concept-word formation) in the formal language of *Grundgesetze*, and I very much doubt that during that period Frege considered abstraction to involve a third sort of genuine concept

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17 Wright (1997), p. 208. Wright is here using the word "concept" in the "usual informal philosophical way".

18 See also *Grundlagen*, §70, where Frege describes the formation both of a relation (he calls it "relation-concept") and of a (simple) concept essentially along the lines of his account, say, in 'Booles rechnende Logik und die Begriffsschrift'.

formation. As far as I can see, there is not even a trace of evidence that he did. There is, however, evidence that, in his view, concepts are just as little creations of the human mind as are thoughts, numbers, truth-values and courses-of-values (cf., e.g., KS, p. 122).

1.4 Identity of content and identity of reference

It is well known that in *Grundlagen* Frege did not yet distinguish terminologically between the sense (*Sinn*) and the reference (*Bedeutung*) of an expression. In this book, he is still indulging in a freewheeling use of the two terms, though he uses them perhaps not always interchangeably. Be it mere accident or for some hidden reason, at least in the course of expounding his context principle he applies the word "Sinn" only to sentences and reserves the word "Bedeutung" for words. By way of contrast, he employs the term "content" for both sentences and words. In his definitions in *Grundlagen* of equinumerosity, the concept of cardinal number (§72), the successor relation (§75), following in a series (§79), etc. Frege uses the word "gleichbedeutend"; this applies also to his tentative contextual definitions of the direction operator and the cardinality operator (cf. *Grundlagen*, §§65, 106). It is clear, however, that "gleichbedeutend" is to be rendered here not as "coreferential" as in his writings after 1890, for instance, in *Grundgesetze* (vol. I, §§3, 10, 33, 144), but rather as "means the same as". Thus, our question is: what does Frege mean in *Grundlagen* when he stipulates that the two sides of an abstraction principle like (1) or (2) shall be "gleichbedeutend" or are to have the same sense or content?

The answer is not immediately to hand. In *Begriffsschrift*, Frege uses, with respect to assertoric sentences, both the terms "conceptual content" ("begrifflicher Inhalt") and "judgeable content" ("beurteilbarer Inhalt").[19] He states indirectly a criterion for the identity of conceptual contents of sentences or judgements: two sentences S_1 and S_2 have the same conceptual content if (and only if) the conclusions which can be drawn from S_1 in connection with certain other sentences $T_1,\ldots,T_n$ can always be drawn from S_2 in connection with $T_1,\ldots,T_n$.[20] Frege does not

[19] In §2 of *Begriffsschrift* (BS), Frege refers to a judgeable content by means of a nominalized phrase: the circumstance that there are houses.

[20] Frege's claim that in a Begriffsschrift there is no need to distinguish between sentences, which have the same conceptual content, is, of course, problematic. But we may ignore this aspect here.

tell us whether he identifies the conceptual content of a sentence with its judgeable content or not. The fact that he does not expressly formulate a criterion for the identity of judgeable contents is, of course, no sign that he tacitly took the criterion for the identity of conceptual contents to apply to judgeable contents as well, that he saw no difference between the two with respect to sentences.

Nonetheless, one might wish to favour the view that for Frege the conceptual content of a sentence coincides with its judgeable content.[21] Following his exposition in *Begriffsschrift*, the former can also be characterized as that part of the content which is relevant when we are to formulate gapless chains of inference.[22] When Frege comes to devise his theory of sense and reference, he divides the judgeable content into thought and truth-value (KS, p. 172; GGA I, p. X; WB, p. 96). But in retrospect he construes the judgeable content primarily as what he then calls the thought (WB, p. 120), that is, as that part of the content of a declarative sentence that can be recognized as true or rejected as false. It is true that Frege does not mention the notion of conceptual content when he casts a brief glance back on the semantic terminology employed in *Begriffsschrift*. We have therefore no clue that he would have been prepared to say that in his theory of sense and reference he split up the conceptual content of a sentence into thought and truth-value. But we may at least assume that he could have explained, without further ado, that the conceptual content of a sentence is that part of its content that can be recognized as true or rejected as false. We may call that part "the cognitive part of the content of a sentence" or simply its "cognitive content". At any rate, I cannot think of any argument that would compellingly show that in *Begriffsschrift* Frege held that two sentences S_1 and S_2 can have the same conceptual content, but different judgeable contents. The possible objection that his introduction of the (undifferentiated) notion of judgeable content besides the notion of conceptual content seems to make sense only if he saw a clear difference between the two notions as applied to sentences has little force. First, it is difficult to find a cogent reason why Frege should have believed he had to operate with two distinct notions of sentence-

21 In §9 of BS, entitled "The function", Frege uses the term "conceptual content" several times. It could always be replaced with "judgeable content", I believe.

22 Frege calls the part of the content, which is the same in a sentence in the active mood and in the corresponding sentence in the passive mood, the *conceptual content*.

content. Second, it is important for his conception of an ideal, formal language to distinguish between judgeable and unjudgeable contents. Moreover, the term "conceptual content" is applied explicitly not only to sentences, but also to singular terms (cf. BS, §8).[23] And I imagine that Frege intended to apply it to functional signs (concept- and relation-signs), too. In sum: due to the lack of textual evidence I do not wish to vouch for the claim that in *Begriffsschrift* Frege tacitly identified the conceptual content of a sentence with the judgeable content. But, on the face of it, I regard it by no means as a less plausible option than the opposite claim. I shall return to this issue shortly when I comment on Frege's criteria for the identity of thoughts.

Let us now turn to *Grundlagen*. The term "conceptual content" does not appear at all in this book, but the term "judgeable content" is used in a few places (cf. GLA, §§70, 74, 104). It is only in Frege's writings after 1890 that both terms have disappeared altogether from his semantic vocabulary. But after 1890 Frege still uses, albeit only rarely, the word "content". The occurrence of "judgeable content" in *Grundlagen* to which I want to draw attention is this: "In the same way with the definitions of fractions, complex numbers and the rest, everything will in the end come down to the search for a *judgeable* content which can be transformed into an equation whose sides precisely are the new numbers" (§104, emphasis M.S.). So, especially by appeal to this remark we can say with some confidence that according to Frege's view in *Grundlagen* a contextual definition presenting itself in the guise of an abstraction principle stipulates that the two sides shall have the same judgeable content, that is, shall have that content in common which in his theory of sense and reference he calls the thought. To be sure, we cannot rule out with absolute certainty that when stipulating that the two sides of an abstraction principle shall be "gleichbedeutend" or shall have the same content or sense, Frege has in mind a content of a very loose kind which coincides neither with the judgeable nor with the conceptual content of a sentence. I consider this possibility rather remote, though. Although Frege's exposition in *Grundlagen* is predominantly informal, he presumably uses the word "gleichbedeutend" in a strict, technical sense when he formulates his definitions.[24] If this is right, we may

[23] Notice, however, that the conceptual content of a singular term is, in Frege's later terminology, the "Bedeutung" not the "Sinn".

[24] There is a terminological analogy in BS with §62 and §65 of GLA. When in §24 of BS Frege defines "The property F is hereditary in the f-sequence" through

conclude that "identity of *content*" of the two sides of an abstraction principle is likewise meant in a technical sense.

It was more than twenty years after the publication of *Grundlagen* that Frege endeavoured to formulate precise criteria for the identity of the senses or thoughts expressed by two sentences. Between 1892 and 1906, we find only a few scattered explanations of the difference or identity of the thoughts expressed by two sentences. In 'Über Sinn und Bedeutung', for example, Frege observes that anybody who did not know that the evening star is the morning star might hold the thought expressed by the sentence "The morning star is a body illuminated by the Sun" to be true, the thought expressed by the sentence "The evening star is a body illuminated by the Sun" to be false (KS, p. 148; cf., e.g., WB, p. 128). By contrast, in the fragment entitled 'Logik' (probably written around 1897), he does not speak in epistemic terms when he comes to discuss the issue of thought-identity. He cites a pair of sentences in the active and passive mood and asserts that from the fact that both express the same thought it follows that it is impossible that one of them should be true while another is false (cf. NS, p. 153).

Now to Frege's two criteria of thought-identity. The first is given in a letter written to Husserl in the year 1906. Frege introduces it by emphasizing that an objective criterion seems necessary for recognizing a thought again as the same, since logical analysis would be impossible without it. He assumes that neither of the two sentences contains a logically evident sense-component. The criterion, henceforth referred to as "CRIT 1", is this (WB, pp. 105f.):

> If both the assumption that the content of [a sentence] *A* is false and that of *B* true, and the assumption that the content of *A* is true and that of *B* false lead to a logical contradiction, and if this can be established without knowing whether the content of *A* or *B* is true or false, and without requiring other than purely logical laws for this purpose, then nothing can belong to the content of *A*, insofar as it is capable of being judged true or false, which does not also belong to the content of *B* [...] Equally, under our assumption, nothing can belong to the content of *B*, insofar as it is capable of being judged true or false, which does not also belong to the content of *A*.

"$\forall d \forall a(F(d) \wedge f(d,a) \rightarrow F(a))$", he also only says that the first expression is to have the same content as the second, without using the attribute "conceptual" or "judgeable". In the same section, he stipulates that the expressions "Δ is the result of applying the procedure f to Γ" and "Γ bears the converse of the f-relation to Δ" are to be taken as "gleichbedeutend". I assume that "same content" and "gleich-bedeutend" are used here in a strict, technical sense.

In a posthumously published piece entitled 'Kurze Übersicht meiner logischen Lehren' [1906] (NS, p. 213), Frege states a different criterion, henceforth referred to as "CRIT 2":

> Two sentences *A* and *B* can stand in such a relation that anyone who recognizes the content of *A* as true must without further ado recognize that of *B* as true and, conversely, that anyone who recognizes the content of *B* as true, must also immediately recognize that of *A* (*equipollence*), where it is presupposed that there is no difficulty in grasping the contents of *A* and *B*.

CRIT 1 is framed in logical, CRIT 2 in epistemic terms. CRIT 1 captures a notion of sense or thought which is akin to the notion of conceptual content, while CRIT 2 captures a notion of sense or thought which seems to be of a finer texture than the first. When we apply CRIT 1 to one of Frege's paradigm cases of abstraction principles, say to (1) or (2), we notice that the two halves of the biconditional would express the same thought. According to CRIT 2, however, the members of the same pair of sentences would presumably express different thoughts, since it seems possible that someone who recognizes one member as true fails to recognize straight away the other as true, for instance, if he or she lacks the concept of direction or of cardinal number.

You will recall my proposal that in *Grundlagen* the stipulation that the two sides of (1) or (2) have the same content is tantamount to the stipulation that they have the same judgeable content. Someone might object that this cannot be correct, because what Frege meant by judgeable content corresponds essentially to the finer-grained notion of thought incorporated in CRIT 2 and not to the coarser-grained embodied in CRIT 1. My answer is this. First, there is no direct evidence that for Frege in *Begriffsschrift* and *Grundlagen* the notion of judgeable content matches essentially with the notion of thought emerging from CRIT 2. But even if there were such evidence— and this would seem to speak against the identification of the conceptual content of a sentence with the judgeable content — we would have no guarantee that in *Grundlagen* he intended to equate the contents of the two sides of an abstraction principle in the sense of CRIT 1. We do not know whether Frege was aware of the tension that appears to exist between CRIT 1 and CRIT 2. He possibly considered the two criteria to be equivalent, especially since he probably formulated them in the same year. Indeed, it is hard to fathom why he should have deemed it necessary to dispose of two distinct notions of thought. If we account for several other remarks Frege made about the difference of sentence-sense, the suspicion grows stronger that he lacked a coherent view of thought-identity and thought-difference. Suffice it to give one example. According to CRIT 1, the thought expressed by "$2^2 = 4$" would be the same as that expressed by "$2 + 2 = 4$" and this would possibly also hold according to CRIT 2. Yet Frege contends explicitly that the two equations express different thoughts (GGA I, p. 7; cf., e.g., WB, p. 235). Returning to the notion of judgeable content, we are bound to state that we do not know exactly how he understood it. We only know from what he says that it is primarily the thought. But unfortunately Frege's notion of thought is not as clear-cut as it should be.[25]

It is time to say a few words about Axiom V or its informal analogue in the present context. In 'Funktion und Begriff' of 1891 Frege stresses that the generality of an equation between function-values "$x^2 - 4x = x(x - 4)$" expresses the same sense as the course-of-values equation "$\dot{\varepsilon}(\varepsilon^2 - 4\varepsilon) = \dot{\alpha}(\alpha.(\alpha - 4))$", but in a different way. By contrast, in §3 of *Grundgesetze* he stipulates only that both sides of Basic Law V are coreferential (*gleichbedeutend*), that is, have the same truth-value. This is confirmed by his remark at the beginning of §10. And in introducing his permutation argument in §10, he says that, on the assumption that $X(\xi)$ is a bijection of all objects (of the domain of his logical system), "$X(\dot{\varepsilon}\Phi(\varepsilon)) = X(\dot{\alpha}\Psi(\alpha))$" is coreferential with "$-\!\!\smile^{\mathfrak{a}}\!\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$", but adds in a footnote: "That is not to say that the sense is the same." In a sense, this remark is trivial, since sameness of reference does not imply sameness of sense. According to Peter Simons (1992, p. 764), it suggests, though, that the sense of "$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$" *is* the same as that of "$-\!\!\smile^{\mathfrak{a}}\!\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$". I cannot see why it should do this. It is true that if Frege believed that the two sides of Basic Law V express the same thought, he would have to gainsay that "$X(\dot{\varepsilon}\Phi(\varepsilon)) = X(\dot{\alpha}\Psi(\alpha))$" and "$-\!\!\smile^{\mathfrak{a}}\!\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$" likewise express the same thought. The obvious

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[25] In the Frege literature of the last decade or so, there is a considerable amount of discussion not only of the problems involved in the semantic stipulations which Frege makes when he introduces his paradigm cases of abstraction principles, but also of the difficulties inherent in his criteria of thought-identity. I do not have the space here to comment in some detail on that discussion, but plan to do so in a monograph on Frege. I refer the reader to Dummett (1991), pp. 168ff.; Hale (1994), pp. 125-130 and Hale (1997); Beaney (1996), pp. 224-234, with the proviso that I disagree with some points especially of Dummett's and Hale's account.

reason is that the two proper names "$X(\acute{\varepsilon}\Phi(\varepsilon))$" and "$X(\acute{\alpha}\Psi(\alpha))$" contain a course-of-values name as a component expression, and the sense of the function-name "$X(\xi)$" is supposed to contribute to the sense of the more complex object name of which "$X(\xi)$" forms a part. Hence, the sense of "$X(\acute{\varepsilon}\Phi(\varepsilon))$", for instance, differs from that of "$\acute{\varepsilon}\Phi(\varepsilon)$" and, consequently, the sense of "$X(\acute{\varepsilon}\Phi(\varepsilon)) = X(\acute{\alpha}\Psi(\alpha))$" differs from that of "$\acute{\varepsilon}\Phi(\varepsilon) = \acute{\alpha}\Psi(\alpha)$". However, nowhere in *Grundgesetze* does Frege contend that both sides of Basic Law V have the same sense. And in the footnote in question he neither affirms nor denies that the senses of "$X(\acute{\varepsilon}\Phi(\varepsilon)) = X(\acute{\alpha}\Psi(\alpha))$" and "$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$" coincide. Clearly, if he affirmed identity of sense in the latter case, he would be committed to the claim that both halves of Basic Law V express different senses. But it is equally obvious that the opposite contention, namely that "$X(\acute{\varepsilon}\Phi(\varepsilon)) = X(\acute{\alpha}\Psi(\alpha))$" and "$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$" have different senses, would not commit him to the claim that both sides of Basic Law V do express the same sense. The upshot is, then, that the footnote reveals virtually nothing about Frege's opinion as to the identity or difference of sense of both sides of Basic Law V. One thing, however, I take to be beyond doubt. If in *Grundgesetze* Frege had been unshakably convinced that both halves of Basic Law V express one and the same thought, he could hardly have believed that it lacked self-evidence. In part II of this essay, I shall say more about Frege's attitude towards Axiom V regarding its lack of self-evidence.

Guillermo Rosado Haddock (1998) criticizes some of the remarks I make in Schirn (1996a) on HP and Axiom V. On p. 258 he says, in his slightly flamboyant style, that I reach the peak of my confusions when I claim "once more that both halves of (T) [i.e. HP] have the same sense, since Frege had claimed that they had the same judgeable content [...], but presumably 'neither claims nor denies that the two sides of Axiom V express the same sense' [...]. In other recent writings, however, Schirn has explicitly said that the two halves of Axiom V, similarities with (T) notwithstanding, do not express the same sense, but only have the same reference." I am at a loss to see any confusion in my account, let alone "a peak of confusions". *First*, I do not say that Frege (explicitly) *claims* that the two sides of HP have the same judgeable content, but rather that he *holds* that view. I admit, however, that I should have made the caveat that there is no conclusive textual evidence that Frege intended to lay down an identity of judgeable contents when he framed a contextual definition of the direction operator in §65 of *Grundlagen*. Yet if he intended to do this — recall that I am still inclined to assume that he did — then we could with perfect justice say that Frege's stipulation amounts essentially to laying down that the two sides of abstraction principle (1) are to express the same thought. Notice that Frege is concerned to *define* contextually the direction operator by using an abstraction principle. After the development of his theory of sense and reference, he says that a constructive or an explicit definition stipulates that the defining expression and the defined expression are to have the same reference and the same sense. *Second*, it is plainly false that I should have said elsewhere that the two halves of Axiom V do not express the same sense, but only have the same reference. It is one thing what Frege says explicitly, another how it is assessed by his interpreters. The fact that Rosado Haddock mixes the two aspects is also obvious from the account he gives in his 1999 paper (see pp. 307ff.). There he speaks of "Schirn's rendering" of Axiom V in Schirn (1995) and contends that it has absurd consequences. Now, in Schirn (1995), p. 16 I say that in *Grundgesetze* "Frege stipulates that both sides of Axiom V shall have the same reference, though he does not explicitly deny that they may have the same sense", and this is undeniably true to the facts. In the sequel, I am exclusively concerned with the unpalatable consequences which follow from the assumption (made by Hodes (1984), but, for instance, also by Sluga (1980), (1986)) that Frege believed in *Grundgesetze* that the two sides of Axiom V do express the same sense. Nowhere did I suggest that for this reason *we* must understand Axiom V as expressing that the two sides have the same reference, i.e. truth-value. Surely, by stipulating only that both sides of Axiom V are to have the same truth-value, Frege faces the serious problem, completely ignored by him, that it remains unfathomable how under that weak interpretation it should bring it about to fix partially the reference of a course-of-values term "$\dot{\varepsilon}\Phi(\varepsilon)$". It was *not* my concern in Schirn (1995) or Schirn (1996a) to discuss this consequence of Frege's stipulation. At any rate, to contend that I advocate the view that *we* should construe Axiom V as expressing an identity of the references of its two sides is wide of the mark. (Notice that for Frege an axiom is a true *thought*, which neither needs nor admits of proof. For the sake of simplicity, I did not strictly adhere to this use of the term "axiom". For a critical discussion of Sluga's and Dummett's views of the nature of Axiom V see my forthcoming book *Begriff und Zahl*.)

2. Abstract and logical objects

Some Frege scholars speak of abstract and logical objects without differentiating between *abstract* and *logical*. Frege himself does not speak of abstract objects, but rather of non-real objects (*nichtwirklichen Gegenständen*) when he deals with those objects which we would call abstract today. Non-real (or abstract) objects in Frege's sense like the axis of the Earth, the equator, the center of mass of the solar system (cf. GLA, p. 35), the numbers and the courses-of-values of functions, introduced for the first time in 'Funktion und Begriff' (1891), are neither accessible to our sensation, intuition, or imagination, nor capable of involvement in physical interaction, nor subjective like ideas, but are non-spatial, non-temporal (i.e. causally inert), though still objective. Probably the best known of the relatively few places where Frege uses the term "logical object" is at the very end of the appendix to *Grundgesetze*: "As the prime problem of arithmetic one may regard the question: How do we conceive logical objects, in particular, numbers? What justifies our recognizing numbers as objects?" (GGA II, p. 265; cf. WB, p. 223). This quotation makes it clear, too, that Frege considered the prime problem of arithmetic to belong first and foremost to epistemology, not to ontology. I hasten to add that without having secured the existence of courses-of-values in the first place, the question concerning our cognitive access to them would be idle, of course. Frege's logical objects — after 1891, they comprise the numbers, which are eventually identified with courses-of-values, the two truth-values the True and the False and the courses-of-values — form only a subset of the set of the non-real (abstract) objects. As regards his conception of non-real objects, it is therefore necessary to distinguish between abstract ones that are non-logical and abstract ones that are logical in nature. Dummett (1991, p. 224) claims that the expression "logical" in the phrase "logical objects" refers to what Frege always emphasized as the distinguishing mark of the logical, namely its generality. According to Frege's criterion of universal applicability, Dummett says, the concept of cardinal number is already a logical one, and need not be defined in terms of extensions of concepts to make it so. I think, however, that a little more has to be said about Frege's conception of logical objects in order to put it in the right perspective.

As a first approximation, I suggest that, for Frege, logical objects are at least all those objects, which are introduced by means of a logical abstraction principle. Admittedly, in his post-*Grundlagen* period he also regards the truth-values the True and the False as logical objects, although he does not introduce them by way of logical abstraction, but rather assumes that every judging person, including the skeptic, is already acquainted with them. In §10 of *Grundgesetze*, however, he identifies the True and the False with special courses-of-values and, hence, with objects introduced via logical abstraction. To be sure, the objective of this transsortal identification is to remedy the referential indeterminacy of course-of-values terms arising from Axiom V, not to establish the logical nature of the truth-values.[26] In part III of this essay, I shall argue that Frege construed the truth-values as the primitive objects of logic. So, let me put them aside now, bearing in mind that my proposed criterion as to what counts as a Fregean logical object does not claim to be exhaustive.

Under what condition is a Fregean abstraction principle a logical one? To account for the distinct frameworks within which Frege pursues his foundational project in *Grundlagen* and in *Grundgesetze*, I suggest we distinguish between two variants of logical abstraction principles. A *logical abstraction principle in a wider sense* I call one whose equivalence relation (Wright calls it "abstractive relation"), denoted on its right-hand side, can be defined in second- or higher-order logic, but whose equivalence between left-hand and right-hand side Frege neither characterizes as a primitive truth or law of logic nor introduces as a logical axiom. A *logical abstraction principle in a narrower sense* I call one whose abstractive relation can likewise be defined in second- or higher-order logic, but whose equivalence between left-hand and right-hand branch Frege considers to be a primitive truth of logic or establishes as a logical axiom.

Many logicians are willing to concede that second-order logic does not enjoy that kind of certainty and security that are characteristic of first-order logic. The reason is that second-order logic makes stronger conceptual and ontological assumptions than first-order logic does. As is well known, Quine is one of the fiercest opponents of higher-order logic; he argues that it is merely set theory in disguise, and, at any rate, he does not consider set theory to belong to logic (see Quine (1970)). Without doubt, the ontology underlying set theory has an enormously wide range, whereas Quine seems to insist that logic should be free of ontology. More recently, Stewart Shapiro has defended the thesis that higher-order logic plays an essential role in the foundations of mathematics. He argues that second-order logic supplies better models of important aspects of mathematics than first-order logic does (cf. Shapiro (1991)). When Frege set about writing *Grundgesetze*, he presumably had no scruples about carrying out his logicist enterprise within a system of second-order logic. In *Grundgesetze*, he regarded set theory (actually, his theory of courses-of-values) as a proper part of logic, but it seems that already before completing this

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[26] Contrary to what Marco Ruffino (1996) claims.

work he had an inkling that the laws governing classes are less basic than the laws relating to concepts (cf. WB, p. 121). Dummett (1991, p. 12) is right when he says that Frege valued his reduction of arithmetic to set theory only so long as he believed it to be a reduction to logic. We do not know whether Frege pursued with genuine interest the rise and development of axiomatic set theory in the first two decades of the 20th century. It may well be that he lacked such interest, because he may have arrived at the conclusion that set theory, even if consistent, cannot provide a logical foundation of arithmetic. Frege considered extensions of concepts or classes to be logical objects only insofar as set theory formed a proper part of logic.

I think it would be a difficult, but worthwhile task to scrutinize Frege's conception of what it means that something belongs to pure logic or that something is formulated in purely logical terms. As far as I see, such scrutiny could only rely on relatively brief remarks scattered throughout a number of his writings. Moreover, I am presently not sure as to whether it would reveal a coherent conception. In *Grundlagen* (p. 117), Frege observes that he has succeeded in reducing one-one correspondence to purely logical terms, to the term "relation", that is. We are further told that a relational concept, like a simple concept, belongs to pure logic. The particular content of the relation is said to be of no concern here; what counts is its logical form. And the truth of whatever can be asserted about this form is analytic and known a priori (GLA, p. 83). Taken at face value, Frege does not distinguish here between logical and non-logical concepts and relations. He seems rather to be claiming that every concept and every relation belongs to logic. Seen in this way, *tree* or *human being* would belong to logic as does the concept of negation, for example. But does Frege really wish to endorse such a conception in *Grundlagen*? It is, moreover, not clear to me what exactly he has in mind when he speaks of the logical form of a concept or a relation, as he does not clarify this by means of an example. What can be said about the logical form of the relation of identity, for instance? That it is a two-place first-level relation? If so, then, for Frege, this is an analytic truth, one we know a priori. What about the statement that identity is an equivalence relation? This, too, seems to be a matter of logical form rather than of content. In the Introduction to *Grundgesetze*, Frege makes basically the same point as in *Grundlagen* concerning the question whether a given expression belongs to pure logic. "Much the same holds for the word 'correspondence' as for the word 'set'; both are today used frequently in mathematics [...] If I am right in thinking that arithmetic is a branch of pure logic, then a purely logical expression must be selected for 'correspondence'. I choose 'relation' for this purpose. Concept and relation are the foundation-stones upon which I erect my structure" (GGA I, p. 3). About sixteen years later, in 'Über die Grundlagen der Geometrie', II (1906), Frege stresses the need to supply a clear characterization of what counts as a logical inference and what is proper to logic, if we are to carry out valid independence proofs. He points out, furthermore, that for this purpose it would be necessary to formulate, in a precise manner, a basic law which one might call an emanation of the formal nature of logical laws (KS, p 321). At the same time, Frege stresses that logic is, despite first appearances, not purely formal. Just as concepts like *point, line, plane* and relations like *lies on, between, congruent* belong intrinsically to geometry, so logic, too, has its own concepts and relations such as negation, identity, subsumption, subordination of concepts for which it allows no replacement. For Frege, this is an unmistakable mark that the relation of logic towards what is proper to it is not at all formal. He concedes, however, that here we find ourselves still in unexplored territory. In my view, all this does not appear to be in line with what Frege says in *Grundlagen*, §70, though.

Consider HP. It is a logical abstraction principle only in a wider sense both in *Grundlagen* and in *Grundgesetze*. Firstly, the relation of equinumerosity is definable in second-order logic as one-one correspondence and the latter can, as Frege puts it (cf. GLA, §§72, 108; GGA I, p. 3) "be reduced to purely logical relationships". Secondly, the question whether HP is a primitive truth of logic is passed over in silence both in *Grundlagen* and in *Grundgesetze*. I conjecture, however, that in neither work did Frege consider it a candidate for being distinguished as a basic law of logic or for being selected as an axiom of his logical theory. In part II of this essay, I shall return to this topic. In any event, both in *Grundlagen* and in *Grundgesetze* Frege derives HP from his explicit definition of the cardinality operator and in this way intends to achieve a twofold purpose: to ensure its requisite analytic or logical character and to lay the foundations for his subsequent derivations of the basic laws of arithmetic. Hence, in neither of the two works does HP enjoy the status of a definition or an axiom.[27] Plainly, only if Frege felt entitled to use HP either as a definition of the cardinality operator (satisfying certain constraints) or as a logical axiom governing it as a primitive expression, could he regard it as a means of *introducing* cardinal numbers as logical objects.

Earlier I pointed out that (CD) involves the introduction of occurrences of the cardinality operator resisting contextual elimination, and that for this reason every attempt to salvage it as a viable definition of "$N_x\varphi(x)$" is doomed to failure. You may also recall my claim that this

27 In *Grundlagen*, HP is only tentatively put forward as a definition, but eventually rejected in this function.

difficulty seems to have gone unnoticed by Frege. Suppose now, for the sake of argument, that in *Grundlagen* Frege succeeded in removing the Caesar objection by making just the right kind of additional stipulation, while adhering to (CD). Seen from the angle of his comments on (CD) in *Grundlagen*, nothing would then prevent him from asserting that cardinal numbers have been introduced as logical objects via (CD), that is, via a logical abstraction principle in a wider sense. It would be guaranteed, from Frege's point of view, that the cardinality operator is defined in purely logical vocabulary, and no more and no less is required to secure the logical nature of cardinal numbers. They could then be acknowledged as logical objects *sui generis*. However, if in *Grundlagen* Frege wished to introduce cardinal numbers by way of HP conceived of as an axiom, he would have to make a convincing case that HP can be acknowledged as a basic law of logic. In this case, it would be insufficient to invoke the fact that one-one correspondence is reducible to purely logical relationships. To be sure, in *Grundgesetze* HP drops out as a candidate for a definitional introduction of logical objects right from the start. It could at best serve as a logical abstraction principle in a narrower sense. After having developed a systematic theory of definition in *Grundgesetze*, Frege rejected contextual definitions altogether, mainly because they infringe the principle of the simplicity of the *definiendum*.

Let us now turn to Basic Law V. Frege could recognize its abstracta as logical objects only if he argued persuasively that it is a fundamental law of logic. It is true that in 'Funktion und Begriff' and in *Grundgesetze* he contends both the unprovability and the logical nature of the possibility of transforming the generality of an identity of function-values into an identity of courses-of-values and vice versa, but it is equally true that he fails to justify this claim. In *Grundgesetze* I, §9 and II, §147, he explains that in logic one has actually made use all along of the possibility of transformation embodied in Basic Law V, although by appeal to the coincidence of functions instead by referring to the equality of courses-of-values. Such an explanation does not carry much weight, however. Frege's appeal to the fact that logicians have long since spoken of the extension of a concept (GGA II, §147) establishes just as little the logical nature of his extensions of concepts. What matters is, in the end, what logicians have actually understood by extensions and how they introduced them. Even if we grant Frege that, whenever logicians use(d) the term "extension of a concept", they tacitly relied on a version of

Axiom V which was restricted to concepts and their corresponding extensions[28], we have no guarantee that their conception of extensions was essentially the same as his. As a matter of fact, the prevailing view among contemporary logicians and mathematicians assumed the extension of a (first-level) concept to consist of the objects falling under the concept. Since logicism in *Grundgesetze* requires that numbers be identified with extensions of concepts, the latter must be of a purely logical character.[29] It follows that in Frege's view the extension of a first-level concept cannot consist of physical objects. Yet this is precisely what the advocate of the predominant conception of extensions has to concede, if the objects falling under the concept are of a physical nature. It is plain that already during the period of *Grundlagen* Frege does not regard agglomerations of physical objects or, more generally, wholes made up of parts as something belonging to logic.

3. Cardinal numbers, extensions of concepts and Julius Caesar

In my opinion, it is fairly obvious that at a number of crucial places in parts IV "The concept of cardinal number" and V "Conclusion" of *Grundlagen* Frege hesitates to put all his cards on the table. It is this hedging attitude which makes it sometimes difficult for the reader to take him at his word or to find out what he really has in mind or aims at when making certain remarks. I venture to surmise that either he had not clearly recognized some fundamental difficulties in his exposition or not yet sufficiently thought through them or, at any rate, not come to grips with them. Here is one bundle of questions that Frege leaves unanswered: Does he regard the "transsortal" identification of cardinal numbers with extensions of concepts as indispensable for his logicist programme?[30] If so, what is it to mean that at the end of *Grundlagen* he

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28 Frege mentions in this context the Leibniz-Boole calculus of logic.

29 For that it is not required that the concept itself is one belonging to logic such as $x = x$, for instance. For Frege, the extension of the concept *horse* is no less a logical object than the extension of the concept $x = x$.

30 Demopoulos (1998, p. 492) suggests in this context: "The project of securing reference to the particular sequence of objects which are *the* natural numbers required the step to equivalence classes since it is unclear how, other than by some such device, one could fashion a definition that would 'comprehend' all applications of the numbers. Were *Grundlagen* expounding a 'pure' theory of number, rather than a theory which aimed to cover both pure and applied statements of number, there would have been no need to introduce extensions." I do not wish to reject this

confesses that he does not attach decisive importance to his introduction of extensions of concepts? In what specific sense is the explicit definition of the cardinality operator supposed to solve the Julius Caesar problem? I shall address these issues in due course.

3.1 The tentative inductive definition of the natural numbers: a spurious Caesar problem

Frege's central insight concerning the logical form of ascriptions of number such as "The number 9 belongs to the concept planet" or "There are exactly 9 planets" in *Grundlagen* is: it contains a predication of a concept. I term this insight *Frege's principle of numerical predication*, PNP for short. Guided by PNP and the result gained by his discussion of Leibniz' definition of the natural numbers (cf. *Grundlagen*, §6), Frege suggests, in §55 of *Grundlagen*, the following inductive definition (ID), where "$N^{n}_{x}F(x)$ is to mean "The number n belongs to the concept F"):

(I) $N^{0}_{x}F(x) = \forall x \neg F(x)$;

(II) $N^{1}_{x}F(x) = \neg\forall x \neg F(x) \wedge \forall x \forall y (F(x) \wedge F(y) \rightarrow x = y)$;[31]

(III) $N^{n+1}_{x}F(x) = \exists x (F(x) \wedge N^{n}_{y}(F(y) \wedge y \neq x))$

In §56, he advances three arguments against (ID) and, unlike the objections he raises to the (partial) contextual definition of "$N_x\varphi(x)$" sustains them all. The first is that we can never decide by means of (ID) "whether the number *Julius Caesar* belongs to a concept, whether this well-known conqueror of Gaul is a number or not [i.e. whether there is a concept F such that Julius Caesar is the number belonging to it]". The second is that we cannot prove with the help of (ID) that a must equal b if $N^{a}_{x}F(x) \wedge N^{b}_{x}F(x)$. The third is that it is only apparent that (I) and (II) define 0 and 1; these definitional clauses fix rather the sense of the

proposal out of hand, but I doubt that it does full justice to the philosophical considerations that accompany Frege's attempted definitions in §§55-68 of *Grundlagen*. I hope that this will emerge at least to some extent from my subsequent discussion.

[31] The *definiens* of (II) could be replaced with the shorter, logically equivalent expression "$\exists x \forall y(F(y) \leftrightarrow y = x)$". For suppose that "$\exists x \forall y(F(y) \leftrightarrow y = x)$" is true and a is an object for which $\forall y(F(y) \leftrightarrow y = a)$ holds. From this follows $F(a) \leftrightarrow a = a$; since $a = a$ holds, we obtain F(a) and eventually $\exists x F(x)$. For an object b distinct from a it follows from $\forall y(F(y) \leftrightarrow y = a)$ that $\neg F(b)$ holds, i.e. a is the only object which falls under the concept F.

second-level predicates "$N^0_x\varphi(x)$ and "$N^1_x\varphi(x)$ (of the statements "$N^0_x F(x)$and "$N^1_x F(x)$ respectively), "but it is not allowed to discern in these 0 and 1 as self-subsistent, recognizable objects."[32]

The first complaint about (ID) seems to be a kind of forerunner of the Caesar objection of §66 of *Grundlagen*. Indeed, §56 is the only place in the book where Frege mentions Julius Caesar. The Roman general is deliberately chosen there as a crude example ("Is Julius Caesar a number?") as is the case with England in §66 ("Is England the same as the direction of the Earth's axis?"). Normally, we would reject both questions out of hand, because to think of Caesar as a number or of England as a direction runs counter to our deeply entrenched intuitions about persons and numbers on the one hand and countries and directions on the other.[33] In any event, it has become common practice in the Frege literature to speak of the Julius Caesar problem and not of the England problem when §66 is under discussion. It goes without saying that the case with which Frege is really concerned in *Grundlagen* is (2). He uses (1) only for the sake of illustration. It is for this reason that I shall transfer the main points of his discussion to the case of equinumerosity (or one-one correspondence) and numerical identity, bearing in mind the essential differences which do exist between (1) and (2), in spite of their similarity. As I said above, the first objection Frege raises in §56 to his heuristic attempt to define the natural numbers inductively appears *prima facie* to be closely linked to what, he thinks, is a compelling ground for abandoning the tentative contextual definition of the cardinality operator (CD) — whence, I believe, comes the phrase "the Julius Caesar problem (objection)" regarding his line of argument in §66. So let us focus on the first objection and see whether it is indeed closely connected or even on a par with the Caesar objection of §66.

If Frege considered numbers to be second-level concepts, as (ID) might suggest, he would be committing a kind of type error by asking whether Julius Caesar is a number. In that case, the concept of number would have to be one under which numbers qua second-level concepts

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[32] In his final brief glance back over the course of his enquiry into the concept of number in *Grundlagen*, Frege considers the third objection to be the main one: (I) and (II) fail to define 0 and 1 separately.

[33] Frege excuses these apparently nonsensical examples, being aware that no one is going to confuse England with the direction of the Earth's axis (or Julius Caesar with the number of planets). The important point is, however, that this is not owing to the tentative definition of the direction operator (or to (CD)).

fall, but Julius Caesar is an object. Thus, when Frege asks whether Julius Caesar is a number he seems to take for granted that the concept of number is of first level. Strictly speaking, the attempted inductive definition (ID) does not permit the formulation of the Caesar objection, because the latter presupposes that numbers are objects, and (ID) obviously does not define the natural numbers as objects, nor was it intended to do this. This becomes even more obvious when we rephrase the *definienda* of (ID) as "There are exactly 0 Fs", "There is exactly one F", etc.[34] Thus, what Frege actually defines are the numerically definite quantifiers. I conclude, then, that the Caesar objection in §56 of *Grundlagen* patently misses its mark and, therefore, has no impact on Frege's project of introducing cardinal numbers as logical objects. By contrast, the worry about Caesar in §66 of *Grundlagen* cannot be dispelled in a similar fashion and, indeed, does affect his foundational programme profoundly. Let me add that the second and third argument against (ID) likewise rest on the assumption, still in need of justification, that numbers are objects.

Quite a few scholars commenting on the Caesar problem in *Grundlagen* tend to mix the Caesar objection in §56 with that in §66 (literally: the England objection), presumably assuming that in both sections Frege faces exactly the same problem.[35] Penelope Maddy (1997, p. 5), for example, contends that Frege required explicit definitions of the numbers to solve the Caesar problem as it arises from (ID) in §56. But this not so. Firstly, I have tried to persuade you that the Caesar problem in §56 is only spurious, not genuine. Secondly, Frege defines the cardinality operator explicitly, because he intends to resolve the Caesar problem laid out in §66 by way of identifying cardinal numbers with equivalence classes. The latter problem, in contrast to that of §56, can be regarded as a genuine one. Notice also that the third objection, which in §66 Frege raises to the attempted contextual definition of the direction operator, has no predecessor in his exposition. Clearly, there is no analogue of (ID) in the case of directions.

In §57 of *Grundlagen,* Frege insists that ascriptions of number be regarded as numerical equations. He apparently wants to get rid of the adjectival use of a numeral and in this way avoid what might cause a

[34] Frege's use of the definite article in ascriptions of number is surreptitious. By smuggling it in, he apparently attempts to conceal from us what clearly is an adjectival use of a numeral.

[35] Cf. Simons (1998), p. 486.

problem for his objectual view of cardinal numbers. Frege's position is ambiguous in this respect, if not incoherent. On the one hand, his remark at the end of §56 that it is not permissible to discern in the phrases "the number 0 belongs to" and "the number 1 belongs to" 0 and 1 as independent objects suggests precisely this: he knew that here a numeral occurs adjectivally as an undetachable part of a second-level predicate and as such performs no referential role. On the other hand, by way of analyzing an ascription of number such as "$N^0_x F(x)$ in accordance with PNP, and by appealing to the fact that the numeral "0" forms only an element of the predicate "$N^0_x \varphi(x)$, he purports to have established (in §57) that in such a statement 0 appears as a self-subsistent object. So, why should he want to explain away ascriptions of number by replacing them with numerical equations when his analysis of the former tallies with his conception of numbers as objects? And why does he highlight PNP as a fundamental insight when he believes that he has to dispense with precisely that type of numerical statement to which PNP is meant to apply? In other words, what motivated Frege to gerrymander his analysis of ascriptions of number, when he insisted, in the same breath, that they ought to be represented through numerical equations? To cut a long story short: my view is that Frege could have accepted ascriptions of number as numerical statements in their own right. Suitably interpreted, they do not threaten, let alone undermine his conception of numbers as objects.[36]

Bearing in mind the heading of §57 of *Grundlagen:* "An ascription of number must be regarded as an equation between numbers", it is tempting to ask why, in his purported attempt to define 0, 1 and n + 1 as objects, Frege did not replace the *definienda* with the corresponding numerical equations. If he had done so, his complaint that the clauses (I) and (II) of (ID) fail to define 0 and 1 separately would equally have applied to the alternative definition (ID*):

(I*) $N_x F(x) = 0 := \forall x \neg F(x)$

(II*) $N_x F(x) = 1 := \neg\forall x \neg F(x) \wedge \forall x \forall y (F(x) \wedge F(y) \rightarrow x = y)$

(III*) $N_x F(x) = n+1 := \exists x (F(x) \wedge N_y (F(y) \wedge y \neq x) = n)$

[36] On these issues cf. Schirn (1996a), pp. 123-135.

Furthermore, Frege could have stressed the need to define "$N_x\varphi(x)$" in the first place before giving correct definitions of "0" and "1" in terms of "$N_x\varphi(x)$". Be that as it may, we could transfer his first objection against (ID) to (ID*): we cannot decide by appeal to (ID*) whether, say, 1 is identical with Julius Caesar, whether he is a number or not. The objection, so construed, would seem to have force, because in the light of Frege's criteria for objecthood or singular termhood[37] we might be willing to accept that (I*)–(III*), unlike the clauses of (ID), are designed to define the natural numbers (contextually) as objects. The reason why (ID*) does not decide whether 1 is identical with Julius Caesar or not, is, plainly, that it fixes the truth-conditions only of equations "$N_xF(x) = 0$", "$N_xF(x) = 1$", "$N_xF(x) = 1 + 1$", etc. just as (CD) does not decide whether Caesar coincides with the number of continents, because it specifies the truth-conditions only of numerical equations of the form "$N_xF(x) = N_xG(x)$".

3.2 The contextual definition of the cardinality operator: the genuine Caesar problem

Earlier, I pointed out that Frege considered it to be a precondition for any methodologically sound introduction of abstract or logical objects to state an adequate criterion of identity for them, that is, to provide a general means to conceive them, to recognize them again as the same, and to distinguish them from any other objects. His attempt to define the cardinality operator through (CD) "$N_xF(x) = N_xG(x) := E_x(F(x),G(x))$" is intended to satisfy this precondition. Frege finds fault with (CD), because it generates the Caesar problem: the identity criterion for cardinal numbers incorporated in (CD), namely

[37] These are the following: (a) The use of the definite article in the singular; here we may ignore typical exceptions like "The whale is a mammal", where the use of the definite article can easily be analyzed away (cf. BS, p. 108; GLA, §§57, 66n, 74n; KS, pp. 169f.); (b) the expression can never stand in the logical place of a predicative expression, though it can be part of one (cf. GLA, §§57, 68n; KS, p. 174); (c) the expression can be used on both sides of the identity-sign or of an equivalent ordinary language expression like "is equal to", "coincides (is identical) with" (cf., e.g., GLA, §§57, 65); (d) the expression is saturated, not in need of supplementation, i.e. it does not contain an argument-place, neither in explicit typographical form as a symbol of a formal language, nor implicitly as an expression of a natural language. For a detailed examination of Frege's criteria see Wright (1983); see also Hale's attempt at framing more reliable criteria by means of which expressions functioning as singular terms may be recognized and distinguished from expressions not so functioning (Hale 1996).

equinumerosity, is powerless to decide whether, say, Julius Caesar is the number of continents, that is, (CD) does not determine the truth-value of an equation of the form "$N_xF(x) = t$" and, hence, does not fix uniquely the reference of the cardinality operator or of a numerical term "$N_xF(x)$".[38] "t" is here an arbitrary singular term which has not the form of "$N_xG(x)$". So understood the Caesar problem is undeniably a semantic problem. Frege himself speaks of a third logical doubt to which (CD) gives rise. He does not yet use the term "semantic". But I suggest that with respect to the third doubt or difficulty "logical" can be rendered as "semantic" and I tend to believe that Frege would have accepted this rendering without further ado, had he been acquainted with a correct use of the word "semantic".

Heck (1997a, pp. 275ff.) has "come to the conclusion" that the Caesar objection in *Grundlagen* poses at least three different, though related, problems. He mentions and discusses only two of them: the first problem is epistemological, the second semantic with "obvious epistemological overtones". I have no idea of what kind the third (or even fourth) problem is supposed to be, but I already disagree with the distinction between an epistemological and a semantic problem concerning Julius Caesar. The observation that Frege raises the Caesar objection against a proposed answer to the question about the mode of how numbers are given to us does not yet justify the claim that this objection poses an epistemological problem. So again, my own view is that the Caesar problem is a semantic problem and not anything besides or over and above that. This applies also in a full sense to the version of the Caesar problem Frege encounters in §10 of *Grundgesetze*.

Why does Frege believe that (CD), if it is to be accepted for the envisaged logical construction of arithmetic, must even decide whether the number of continents is identical with, say, Julius Caesar or the planet Mars? The correct answer is presumably this: He believes that,

[38] Several pertinent remarks which Frege makes in §§55-68 of *Grundlagen* support my assumption that he is primarily concerned to fix uniquely the "Bedeutung" of a numerical term "$N_xF(x)$" in his later technical use of the word "Bedeutung". Frege does not comment on a case like "The number of planets = 6 + 3". I suppose, however, that he was aware that the criterion of identity embodied in (CD) could be applied to such an instance of "$N_xF(x) = t$" only if it were legitimate to replace "t" with a term of the form "$N_xG(x)$. Incidentally, it would be interesting to know how, in view of his critical discussion of the tentative contextual definitions, Frege would have assessed sentences like "The number of planets = the direction of line a" or "The number of planets = the shape of the triangle d".

because in *Grundlagen* he tacitly takes the domain of objects to be both homogeneous and all-inclusive. By "homogeneous" I mean that Frege does not distinguish between categories or types of objects. Every object, be it the number of churches in Rome in January 2002 or the class of continents or Julius Caesar, can be the argument of every first-level function (of every first-level concept or of every first-level relation respectively). By way of contrast, Frege distinguishes types of functions according to the kind of admissible arguments as well as the number of empty places in the corresponding function-names. To my mind, the matter presents itself in a similar fashion in *Grundgesetze*, but since it is set out there in a formal framework, it appears in much sharper outline than in *Grundlagen*. The issue as to whether the domain of the first-order variables of Frege's logical theory in *Grundgesetze* embraces all objects whatsoever or only the two truth-values and courses-of-values is a controversial one in the Frege literature. Despite the formal setting provided in *Grundgesetze*, Frege fails to specify explicitly the first-order domain and, hence, fails to improve on *Grundlagen* in that respect. In part III of this essay, I shall argue that some remarks Frege made in *Grundgesetze* suggest that, contrary to the way in which he actually proceeds in §§10 and 31, he takes the first-order domain of his logical theory to be all-encompassing.

However this may be, (CD) does not even provide us with a means to determine the truth-value of a numerical equation such as "The number of planets = 1" or "The number of planets = the tenth element of the ω-sequence $\langle 0, 1, 2, \ldots \rangle$". For it is obvious that the definitions "$0 := N_x(x \neq x)$" and "$1 := N_x(x = 0)$", set up in §§74, 77 of *Grundlagen*, presuppose that "$N_x\varphi(x)$" has a determinate reference, and that means in the context of *Grundlagen*: they presuppose a prior irreproachable definition of "$N_x\varphi(x)$". If we already had a definition of the concept *n is a number* (henceforth abbreviated by "N(n)") satisfying Frege's requirement of sharp delimitation, so that it is determined for every object of the domain whether it falls under that concept or not, we could stipulate: if t is not a number, then "$N_xF(x) = t$" is false (since in that case $N_xF(x)$ cannot be identical with t); if t is a number and if it is given to us appropriately, then (CD) will settle the truth-value of "$N_xF(x) = t$". At this stage of the inquiry, however, the indeterminacy arising from (CD) cannot be removed by setting up the definition "$N(n) := \exists\varphi(N_x\varphi(x) = n)$", since "$N_x\varphi(x)$", forming a part of the *definiens*, has not yet acquired a determinate reference. In other words, in order to apply the definition of "N(n)", we would first have to know, in each case, whether "$n = N_xF(x)$" is true or false. It is worth noting that even in the case of numerical equations of the form "$N_xF(x) = N_xG(x)$", HP does not, for every instance, determine its truth-value. Take the equation (p) "$N_x(x = x) = N_xFN(x)$", where "FN(x)" is to abbreviate the predicate "finite number". As Boolos shows (1987, p. 16), (p) is an undecidable sentence in the formal system FA; it is true in some models of FA, and false in others. Frege's remarks on Cantor in §86 of *Grundlagen* suggest, though, that he would have considered (p) to be false.

Before making his final attempt to define "$N_x\varphi(x)$", Frege considers a possible solution of the problem that by appeal to (CD) we cannot determine the truth-value of a sentence "$N_xF(x) = t$". One might be tempted to stipulate that the object t is a cardinal number if it is introduced by means of (CD). Frege dismisses the proposal on the grounds that we would be treating the way in which t is introduced as a property of t, which it is not.

It is essential for Frege's foundational project, resting on a classical logic with a classical semantics as it does, to ensure that every concept and every relation (more generally: every function) of the formal theory has sharp boundaries as well as to secure a reference for every well-formed expression, especially for every well-formed formula of his Begriffsschrift. Only in this way, he thinks, can he guarantee the validity of the laws of classical logic in his formal theory, in particular, the validity of the law of excluded middle; only thus can he ensure the general validity of the semantical principle of bivalence. There can hardly be any doubt that his investigation in *Grundlagen* is guided by these methodological principles. We must, of course, bear in mind in this connection that several characteristic marks of Frege's logical theory in *Grundgesetze* are still absent in the formal theory underlying the informal considerations of *Grundlagen* or had not yet fully crystallized at that time. I may mention only two well-known facts: (a) Frege had not yet drawn a terminological distinction between the sense and the reference of a sign; (b) he had not yet introduced the truth-values as the references of assertoric sentences or special object-names.

I have just suggested that already in *Grundlagen* Frege sets up the requirement of the sharp delimitation of a concept or a relation. This has been doubted, for example, by Heck (1997a, pp. 299f.), but the doubt is groundless. When Frege comes to define the number 0 in §74 he observes: "All that can be demanded of a concept from the point of view of logic and with an eye to rigor of proof is its sharp delimitation, that, for every object, it be determined whether or not it falls under the concept." Clearly, this condition must be satisfied by a concept like *the number of planets* = ζ or ξ *is a number*, if it is to be employed in logic. If it lacked a determinate truth-value as its value for the argument, say, Julius Caesar, Frege would stigmatize it as a pseudo concept. Every sentence in which its name occurs would lack a truth-value.[39] Remember in this context my claim that in *Grundlagen* Frege tacitly takes the first-order domain to be all-embracing.

One final comment on (CD) at this stage. It seems that, from Frege's point of view, (CD) fails to fix completely or uniquely the reference of the cardinality operator, even if the Caesar objection could be removed by appeal to an additional stipulation. Even in that case Frege would, I believe, concede that (CD) effects only a partial determination of the reference of a term "$N_xF(x)$". The reason is that his demand of providing explanations of all other (relevant) statements about cardinal numbers is set up quite independently of the emergence of the Caesar problem. In fact, Frege formulates the demand in §65 of *Grundlagen* just before he brings up this problem. By contrast, in §10 of *Grundgesetze* his proposal to determine the values of all (primitive) first-level functions for courses-of-values as well as for all other arguments, is presented as the appropriate response to his diagnosis that Axiom V fails to determine completely the references of course-of-values terms, i.e. that it creates a version of the Caesar problem.[40]

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[39] Cf. in this context *Grundgesetze*, vol. II, p. 74. "If, e.g., the relation *greater than* is not completely defined, then it is likewise uncertain whether a quasi-conceptual construction obtained by partly saturating it, e.g., *greater than zero* or *positive*, is a proper concept. For it to be a proper concept, it would have to be determinate whether, e.g., the Moon is greater than zero. We may indeed stipulate that only numbers can stand in our relation, and infer from this that the Moon, not being a number, is also not greater than zero. But with that there would have to go a complete definition of the word 'number' and that is just what is usually lacking."

[40] The indeterminacy problem arising from Axiom V is, in Frege's own words, that the (informal) stipulation in §3 of *Grundgesetze*: "I use the words 'the function $\Phi(\xi)$ has the same course-of-values as the function $\Psi(\xi)$' generally to denote the same as the words 'the functions $\Phi(\xi)$ and $\Psi(\xi)$ have always the same value for the same argument'" fails to determine completely the reference of a course-of-values

3.3 The cardinality operator explicitly defined

In §68 of *Grundlagen*, Frege eventually defines the number which belongs to the concept F as the extension of the (second-level) concept *equinumerous with the concept* F. (Henceforth, I use "#" as a class-forming operator.)

(ED) $N_xF(x) := \#\varphi(E_x(\varphi(x),F(x)))$.

He presumably construes the expression "the extension of the concept...", forming a part of the *definiens*, as an indefinable primitive, just as he does later in *Grundgesetze*. If this is correct, it would have been incumbent upon him to elucidate it semantically or to fix its content (its sense and its reference according to his theory after 1891) in a different, but likewise non-definitional manner, in order to be entitled to use it in (ED). And, of course, in the light of his logicist aims, he would have to justify that the expression "the extension of the concept..." belongs essentially to logic or in other words: that extensions of concepts are logical objects.[41]

HP, taken jointly with (ED) and the definitions of "0", "1", etc., provides a means of determining the truth-value of any identity-statement of the following six types:

(1) $N_xF(x) = N_xG(x)$

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term "$\dot{\varepsilon}\Phi(\varepsilon)$". While in *Grundlagen* Frege attempts to resolve the referential indeterminacy of the cardinality operator by defining it explicitly, in *Grundgesetze* he must take an entirely different route to remove the indeterminacy of the course-of-values operator. The latter is one of the primitive signs of the formal language of *Grundgesetze* and can therefore not be defined, but at best only be elucidated.

[41] In his list of Frege's posthumous works, H. Scholz mentions an early definition of the extension of a concept (cf. Schirn (1976), vol. I, p. 95). This is only a vague reference, though. I see therefore no need to withdraw my conjecture that in *Grundlagen* Frege considered "the extension of the concept..." to be a primitive expression. In *Grundgesetze*, the course-of-values operator is the only primitive function-name, which is not introduced by means of a semantic elucidation. Its sense and its reference are supposed to be fixed (though only partially) via an axiom, namely via Axiom V. Frege has never told us that this procedure runs counter to his own principles. I shall say more about this topic in part II of this essay. If Frege had been able to devise a sound elucidation of the course-of-values operator, that is, one, which did not rest on a presupposed acquaintance with courses-of-values, then, from his point of view, the referential indeterminacy of course-of-values terms would probably not have arisen at all. In that case, he could even have defined the predicate "a is a course-of-values" ("CV(a)") modelled upon his definition of "n is a number" in §72 of *Grundlagen*: $CV(a) := \exists\varphi(\dot{\varepsilon}\varphi(\varepsilon) = a)$.

(2) $N_xF(x) = \#\varphi(E_x(\varphi(x),G(x)))$

(3) $\#\varphi(E_x(\varphi(x),F(x))) = \#\varphi(E_x(\varphi(x),G(x)))$

(4) $N_xF(x) = n$

(5) $n = \#\varphi(E_x(\varphi(x),F(x)))$

(6) $n = m$

An equation of type (4), (5), or (6) is capable of being reduced to one of type (1), (2), or (3), and the truth-conditions of the latter three are determined through the right-hand side of HP, or, spelled out more fully, through the one-one correlation of the objects falling under F with those falling under G: $\exists R(\forall x(F(x) \rightarrow \exists y(R(x,y) \wedge G(y))) \wedge \forall y(G(y) \rightarrow \exists x(R(x,y) \wedge F(x))) \wedge \forall x\forall y\forall z((R(x,y) \wedge R(x,z) \rightarrow y = z) \wedge (R(x,z) \wedge R(y,z) \rightarrow x = y)))$.

Thus, by virtue of (ED), Frege succeeds in extending the range of applicability of the originally proposed criterion of identity for cardinal numbers. Nevertheless, the criterion so extended lacks the required unrestricted generality. Take, for example, the two equations "$N_x(x \neq x) = \#x(x \neq x)$" and "$N_x(x \neq x)$ = Julius Caesar". The first cannot be transformed into an equation of type (1), since "$\#x(x \neq x)$", unlike "$\#\varphi(E_x(\varphi(x),x \neq x))$", denotes the extension of a *first*-level concept. Hence, HP is powerless to decide whether "$N_x(x \neq x) = \#x(x \neq x)$" is true or false. Although Frege does not comment on such a case, he seems to assume that (ED) removes the referential indeterminacy of the cardinality operator once and for all. If we really do know what the extension of a concept in general is — and Frege takes that for granted, albeit without any justification — then we ought to be able to distinguish the extension of a first-level concept from the extension of a second-level concept, and the latter from Julius Caesar. Consequently, we are justified in assigning the truth-value *false* to both equations. Recalling Frege's discussion of (CD), it seems that he might have considered the following alternative strategy: Once N(n) has been defined without fallacy, we can solve the Caesar problem by appeal to HP and the definition of N(n).[42]

[42] Julius Caesar is not a cardinal number, because there exists no concept φ such

Be this as it may, both strategies are equally unsatisfactory. The problem is by no means solved, but only postponed. For we do not know whether Julius Caesar is a number unless we know whether or not he is an extension. When Frege defines cardinal numbers as extensions of concepts in *Grundlagen*, he has stated identity-conditions only for various kinds of equivalence classes, of equinumerosity and, for the sake of illustration, of parallelism and geometrical similarity. And we have seen that his assumption regarding extensions of concepts in general lacks foundation.

It is, however, fair to say that in §73 of *Grundlagen*, in the course of adumbrating his proof of HP, Frege seems to rely tacitly on an abstraction principle that states for second-level concepts and their extensions what Axiom V states for first-level concepts and their extensions. In order to prove HP, he has to show, according to (ED), that $E_x(F(x),G(x)) \rightarrow \#\varphi(E_x(\varphi(x),F(x))) = \#\varphi(E_x(\varphi(x),G(x)))$. That is to say: he has to prove that "under this hypothesis" the following two sentences hold generally:

(a) $E_x(H(x),F(x)) \rightarrow E_x(H(x),G(x))$ and (b) $E_x(H(x), G(x)) \rightarrow E_x(H(x),F(x))$.

So, Frege is converting here the statement that the extensions of two special second-level concepts coincide into the statement that these concepts are coextensional (see also Heck (1995), p. 130). We might thus presume that in introducing extensions of second-level concepts he had in mind the following third-order abstraction principle:

$$\#f(M_\beta(f(\beta))) = \#f(N_\beta(f(\beta))) \leftrightarrow \forall f(M_\beta(f(\beta)) \leftrightarrow N_\beta(f(\beta))).$$

Yet, if this is so, it would remain obscure why he does not refer explicitly to this principle when he comes to introduce extensions of second-level concepts. As against this, some might wish to argue that in *Grundlagen* Frege must have been aware of the necessity of laying down such a principle as an axiom of a formal theory whose first-order domain comprises extensions of second-level concepts.[43]

that the number belonging to φ is Julius Caesar.

[43] It may well be that in introducing extensions in *Grundlagen* he had already in mind the transformation of the mutual subordination or general equivalence of first-level concepts into an identity of extensions, and vice versa, though it seems unlikely to me that he already thought of the more general transformation, embodied in Axiom V, concerning functions and courses-of-values. Yet if this is so, it would be hard to understand why he assumes (perhaps with certain scruples) that we know what extensions are instead of introducing at least extensions of first-level concepts by appeal to the transformation just mentioned.

3.4 Why Frege introduced extensions in Grundlagen

Marco Ruffino has challenged the view, espoused by several Frege scholars including myself, that in *Grundlagen* Frege gave his explicit definition of the cardinality operator to resolve the Julius Caesar problem arising from his third objection to (CD). Ruffino holds that, according to Frege, the identification of cardinal numbers with extensions of concepts was "absolutely essential" for his logicist programme, not only in *Grundgesetze*, but already in *Grundlagen* (see Ruffino (2000), p. 240f.). In his paper 'Logicism: Fregean and Neo-Fregean' (1998), p. 182 he writes (cf. Ruffino (2000)):

> First, as it seems to me, the primary role of the Julius Caesar problem in Frege's course of thought in GLA is not to point out the residual indeterminacy unsolved by the contextual definition of §§63-5, but rather to call attention for [to] the fact that this definition fails to make evident the logical nature of numbers [...] Second, given the privileged status of extensions as logical objects in Frege's thought, he would have to come to the definition of numbers as extensions anyway — quite independently of the considerations about the Julius Caesar problem. For, otherwise, he would owe us an explanation of the logical status of numbers and why they are on a par with extensions without being themselves extensions. The assumption that Frege opted for a definition of numbers *just* to solve the Julius Caesar problem [...] is simply incorrect.

Related to *Grundlagen*, this assessment strikes me as unwarranted. Here are my reasons:

(i) Ruffino will certainly agree that Frege feels obliged to jettison (CD), because it generates the Caesar problem. However, there is no clue whatsoever in *Grundlagen* backing the assumption that for Frege the primary role of that problem is to draw attention to the fact that (CD) fails to make evident the logical nature of cardinal numbers.[44] Similarly, in *Grundlagen* we do not find direct textual evidence that Frege identified cardinal numbers with equivalence classes of concepts under one-one correspondence, because he was convinced from the outset that only in this way could he secure their logical nature. On the contrary: if we adhere strictly to the line of argument in §§66-69, 107 of *Grundlagen*, there is every indication that Frege would have preferred

[44] This problem simply does not play a primary and a secondary role. It consists in the fact that (CD) does not uniquely fix the reference of numerical terms of the form "$N_xF(x)$".

not to invoke extensions of concepts, had he been able to contrive a sound solution of the Caesar problem in connection with (CD), say, by devising an appropriate additional stipulation which was consistent with (CD). It is therefore by no means the problem that in the light of (CD) the logical nature of cardinal numbers appears to be unfounded or doubtful, but it is exclusively the referential indeterminacy of "$N_x\varphi(x)$" and, consequently, the threat that such indeterminacy is transferred to every expression later to be defined in terms of "$N_x\varphi(x)$", which motivate Frege to abandon (CD) and to pursue a new course by putting forward the explicit definition in §68 of *Grundlagen*.

(ii) It is at least misleading to claim that in *Grundlagen* extensions of concepts have a privileged status as logical objects. Frege introduces them rather abruptly and ad hoc, sparing himself the trouble of justifying their presumed logical character; he does not even touch upon this crucial issue. If he was determined to define cardinal numbers as special extensions of concepts in any event, quite independently of the impact of the Caesar problem, as is claimed by Ruffino, then the reader of *Grundlagen* should feel he has been led up the garden path. Frege mentions the motive for introducing extensions of concepts in §68 unequivocally: (CD) does not enable us to gain a sharply delimited concept of cardinal number. Yet such a concept is needed for laying the logical foundations of arithmetic, and Frege gains it only if he succeeds in removing the Caesar objection, either by making a plausible additional stipulation consistent with HP or by changing the definitional strategy. His summary in §107 of the main results of his enquiry into the concept of number buttresses my interpretation and suggests that the one proposed by Ruffino defies credibility.

> We would not be able to judge on the basis of such a definition whether an equation is true or false if only one side of it is of this form [i.e of the form "The number of Fs = the number of Gs"]. This caused us to give the definition: The number which belongs to the concept F is the extension of the concept "concept equinumerous with the concept F" [...] In doing this, we presupposed that the sense of the expression "extension of the concept" is known. This way of overcoming the difficulty will probably not meet with universal approval, and some will prefer to remove that doubt in another way. I do not place decisive weight on the introduction of the extension of a concept anyway.

The difficulty to which Frege alludes is, to all appearances, that by means of (CD) we do not attain a sharply defined concept of cardinal number. As to the phrase "this way of overcoming the difficulty", I suggest that he intends to refer to (ED) *together with* his assumption that it is known what the extension of a concept is. How the difficulty could be resolved, apart from pursuing the method which Frege actually applies, remains in the dark, though. The only thing we seem to know is that it would have to be a solution without the use of extensions of concepts. This at least is suggested by Frege's puzzling remark "I do not place decisive weight on the introduction of the extension of a concept anyway".

Several issues in connection with Frege's transition from (CD) to (ED) require further comment. Here is one. When at the beginning of §68 of *Grundlagen* he says: "Since in this way we cannot attain a sharply delimited concept of cardinal number", it is not entirely clear whether here he has in mind the cardinality function $N_x\varphi(x)$ or the concept *n is a number* or both.[45] That depends mainly on what he means by "in this way". If he intends to refer solely to (CD) and the third logical doubt it raises, then the first option suggests itself. But if he intends to appeal to the two abortive attempts to resolve the Caesar problem just before he introduces extensions of concepts — first by considering the definition of the concept of cardinal number in terms of $N_x\varphi(x)$ (at the end of §66), and second by stipulating that t *is* a cardinal number if it is introduced by means of (CD) (at the beginning of §67) — then the second option would be the right one. Finally, if Frege wishes to refer to his entire line of argument in §§66-67, then we should probably vote for the third option.

It seems that in *Grundlagen* Frege does not distinguish terminologically, at least not explicitly, between a concept and a mere function, which is not a concept.[46] When we transform a statement of

[45] When in §63 of *Grundlagen* Frege writes: "As against this, it must be noted that for us the concept of cardinal number has not yet been fixed, but rather is to be determined by means of our definition", he clearly refers to (CD). However, (CD) is to define the cardinality function, not the concept of cardinal number. The remark thus seems to show that Frege calls not only N(n), but also $N_x\varphi(x)$ the concept of cardinal number.

[46] In *Grundlagen*, Frege uses the word "concept" perhaps not always in a clear-cut or uniform way. When he explains: "A concept is for me that which can be predicate of a singular judgeable content" (GLA, §66, fn. 2), it is not quite clear

equinumerosity into a numerical equation, what we attain is not, in a strict sense, a new concept (*x is a number*), but rather a new function, namely the cardinality function.[47] If at the outset of §68 Frege referred to the cardinality function, he would be saying, in the light of his theory of functions after 1891, that (CD) does not supply us with a function whose value for every first-level concept as argument would be determined. To be sure, the indeterminacy of the reference of "N(n)" is only a consequence of the indeterminacy of the reference of "$N_x\varphi(x)$", since the first expression has to be defined with the help of the latter and not the other way around. It might well be that at the beginning of §68 Frege wants to give us to understand that the Caesar objection raised to (CD) thwarts his plan of stating an unobjectionable definition of "N(n)" by means of "$N_x\varphi(x)$" and has therefore to be resolved. The heading of §66 "In order to obtain the concept of cardinal number, one must fix the sense of a numerical equation" could be interpreted along these lines. However this may be, it is first and foremost the cardinality function or the cardinality operator that has to be obtained by fixing the sense of a numerical equation.

To summarize: It is of immense importance for the viability of Frege's foundational programme, to remove the referential indeterminacy of "$N_x\varphi(x)$" resulting from his third objection to (CD). The indeterminacy would infect the expressions "N(n)", "0", "1", if these are defined in terms of "$N_x\varphi(x)$", as is Frege's intention. If he were primarily concerned with the sharp delimitation of N(n), as his remark at the beginning of §68 might suggest, then we should expect him to formulate in the same paragraph a definition of "N(n)" satisfying the condition of sharp delimitation. But Frege does not and, of course, cannot take this route. Rather, in a final attempt, he defines the

whether a concept is here conceived of as a part of a judgeable content or rather as the reference (Bedeutung) of a predicate as is the case in his later semantic theory (cf., e.g., KS, p. 172). When he says, however, that a general concept-word *denotes* a concept, he seems to be regarding a concept as the reference of a concept-word. In *Begriffsschrift* as well as in 'Booles rechnende Logik und die Begriffsschrift', Frege seems to have understood by a concept primarily a simply unsaturated part of a thought, that is what after 1891 he calls the sense of a concept-expression. However this may be, by far most of the uses of "concept" in *Grundlagen* agree with Frege's later view that a concept is the reference of a predicate. This applies especially to §§46–54 and chapters IV and V. In what follows, I shall understand "concept" in the context of *Grundlagen* always in this sense.

[47] Cf. Frege's claim in §64 of *Grundlagen* that by transforming parallelism between lines into an identity of directions we obtain a new concept.

cardinality operator explicitly and seems to believe that this already solves the Caesar problem without having to invoke the definition of "N(n)" by means of "$N_x\varphi(x)$" set up later in §72. To be sure, the Caesar problem as described by Frege in §66 emerges only in connection with an abstraction principle of the form $Q(\alpha) = Q(\beta) \leftrightarrow R_{eq}(\alpha, \beta)$. Such a principle serves to introduce a new functional expression or singular-term-forming operator, but not a new predicate. Moreover, it is undeniably the explicit definition of "$N_x\varphi(x)$" and not the one of "N(n)" which plays the definitional key role in the logicist programme set out in *Grundlagen*. It is the first definition and not the second which implies HP, and HP itself is the pivot of the formal derivations of basic theorems of cardinal arithmetic. The explicit definition of "$N_x\varphi(x)$" (ED) does not play any formal role in the proofs of those theorems once HP has been deduced from it.

I have just defended the claim that in *Grundlagen* Frege introduces extensions of concepts, because he intends to solve the Caesar problem, mooted but left unsolved in §66, by defining cardinal numbers explicitly as equivalence classes of equinumerosity. It is time to add one more brushstroke to the picture. When Frege rejected (CD) in the light of the Caesar problem, he must have realized the need to guarantee the analytic status of HP in order to keep logicism intact. Once HP was divested of its role as a (tentative) definition, its analytic status was no longer safe. In particular, the appeal to the fact that its right-hand side is couched in purely logical terms (i.e. that one-one correspondence "is reducible to purely logical relationships") could no longer secure the assumed logical nature of cardinal numbers. Given that Frege construed HP as an indispensable basis for the deduction of the fundamental laws of number theory, he had *in principle* just two options to reestablish it as an analytic or a logical truth: to argue that it is a primitive law of logic, and thus a proper candidate for being singled out as a logical axiom, or to put forward an explicit definition of the cardinality operator which implies HP. Clearly, only on condition that the *definiens* is framed in purely logical terms can he recognize HP as an analytic truth by deriving it from the definition according to purely logical rules of inference. Seen from his angle, Frege has good reasons to choose the second option in *Grundlagen*. First, even if he considered the first option to be viable, the obstinate Caesar problem would not loosen its grip. Rather, it would bother him in the same way as before. Second, I have already voiced doubts that Frege would have been prepared to accept HP as a primitive law of logic. In sum, (ED) is intended not only as a means of overcoming the Caesar problem, but also of rendering possible that HP be salvaged as an analytic truth.

3.5 A puzzle about extensions

What are we to make of Frege's puzzling remark "I do not place decisive weight on the introduction of the extension of a concept anyway"? In my view, it mirrors only his discordant attitude towards the role of extensions in *Grundlagen*, and is at odds with what he actually does and says elsewhere in the book. Thus, it strikes me as curious that, on the one hand, he stresses the key role of extensions of concepts for the envisaged definitions of fractions, irrational and complex numbers and, on the other, contends that they can eventually be dispensed with.[48]

In §104 of *Grundlagen*, Frege deals briefly with fractions, irrational numbers and complex numbers. Just as in the case of cardinal numbers, here, too, "everything will in the end come down to the search for a judgeable content which can be transformed into an equation, whose sides precisely are the new numbers. In other words, we must fix the sense of a recognition-judgment for such numbers. In doing so, we must not forget the doubts raised by such transformations, which we discussed in §§63-68. If we follow the same procedure as we did there, then the new numbers are given to us as extensions of concepts."

Frege's preferred strategy for the introduction of the new numbers as logical objects appears to be this. In a first step, he has to contrive a suitable equivalence relation R for, e.g., the case of the real numbers, which can be defined in purely logical vocabulary. In a second step, the real numbers are tentatively introduced via a logical abstraction principle in a wider sense, namely by transforming $R_{eq}(\alpha,\beta)$ into an identity of real numbers $\Sigma(\alpha) = \Sigma(\beta)$ and by presenting this transformation as a contextual definition of the Σ–operator. (For convenience, I refer to the hypothetical abstraction principle for the reals "$\Sigma(\alpha) = \Sigma(\beta) \leftrightarrow R_{eq}(\alpha,\beta)$"

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48 I suppose that Frege does not mean to give to understand that he does not attach importance to a *logical* foundation of arithmetic when he says that he places no decisive weight on bringing in extensions of concepts at anyway. The goal of logicism seems to be out of the question for Frege both in *Grundlagen* and in *Grundgesetze*.

as "AR".) In doing this, one has to be aware of the logical doubts to which such transformations give rise. In particular, from the analogous case of cardinal numbers and equinumerosity it is to be expected that the criterion of identity for real numbers encapsulated in AR proves to be too narrow, does not determine the truth-value of an equation, say, "$\Sigma(\alpha)$ = Julius Caesar" and, hence, does not fix uniquely the reference of "Σ". To avoid that the logicist programme is put in jeopardy, the difficulty has to be removed. The supposed solution is that in a third and final step the real numbers are explicitly defined as extensions of concepts, or more specifically, as equivalence classes of R.

So, if we rely on Frege's sparse remarks in §104 of *Grundlagen*, then his attempt to define fractions, irrational numbers and complex numbers contextually using second- or higher-order abstraction would lead to a whole family of Caesar problems all of which are supposed to be resolved by framing appropriate explicit definitions for these numbers. However, we have no clue as to how far the analogy between the explicit definition of the cardinality operator (ED) and the envisaged explicit definitions of the new numbers was supposed to go. Did Frege think he had to constrain the explicit definition of, say, the operator "Σ" in such a way that it implied the equivalence AR, whatever it might have been in the end? Did he believe that AR could play a key role in the formal derivations of fundamental theorems of analysis similar to the one HP was designed to play in the formal proofs of the basic laws of number theory? Suppose he believed in such a key role of AR. In this case, he would have had to impose exactly that constraint on the explicit definition of "Σ" I just mentioned. For unless he considered AR, perhaps contrary to expectation and in contrast to HP, to be a primitive law of logic, how else could he have secured the requisite analytic or logical nature of AR, once its definitional status was abandoned in the light of the Caesar problem for real numbers? Of course, we must also ask why Frege suggested that the new numbers should be introduced at all along the lines of his introduction of cardinal numbers. Thus, what made him believe that in these cases, too, he should start with a tentative contextual definition? One possible answer is, I think, that he attached crucial importance to first laying down a general criterion of identity for new logical objects whenever they were going to be introduced. The fact that in §104 Frege sets up the requirement of first fixing the sense of a recognition-judgment for the new numbers supports this view.

The foregoing considerations reinforce my conjecture that Frege could defend his disclaimer concerning extensions only if he offered a solution of the Caesar problem which does not rest on the introduction of extensions. Of course, such a solution would have to ensure that the assumed logical nature of cardinal numbers (and of fractions, complex and irrational numbers) remains unscathed, that is, Frege would have to advocate logicism without classes. It is, however, far from clear how he could accomplish this within the setting of *Grundlagen*. So let us examine more closely which options to remedy the Caesar problem without invoking classes there are and whether any of them proves to be viable. If I am right, there are at least three options.

(1) The first option is to formulate an explicit definition of the cardinality operator without recourse to extensions of concepts. In a famous footnote to §68 of *Grundlagen*, Frege expresses the belief that in (ED) the words "extension of the concept" could be replaced with the word "concept". For brevity, I refer to this substitution by means of "SUB". SUB yields an alternative explicit definition of the cardinality operator. Some scholars have proposed that there seems to be an immediate connection between Frege's remark that he places no decisive weight on the introduction of extensions of concepts and SUB. Sluga (1980, p. 142), for example, conjectures that from Frege's remarks in the footnote it appears that the object denoted by an expression of the form "the concept F" would not be the extension of a concept, and that here Frege contemplated the possibility of identifying the numbers with such objects [with which objects?] rather than with extensions of concepts. Benacerraf (1981, p.61f.) argues in a similar vein when he comes to interpret the footnote to §68 and Frege's attempt to play down, by an incidental remark, the importance he attaches to the introduction of extensions of concepts. He writes: "Thus in precisely this context — the one most critical for determining whether he required definitions to preserve reference — Frege backs off and allows that different definitions, providing different referents (not 'bringing in the extensions of concepts at all') might have done as well. […] The moral is inescapable: Not even reference needs to be preserved." However, I share his opinion just as little as I do Sluga's.[49] First, it is

[49] According to Simons (1992, p. 755), Frege says in the footnote to §68 that he thinks it would be possible to do without extensions of concepts. But this is simply false; see my points below. Concerning Frege's introduction of objects of a quite special kind in 'Über Begriff und Gegenstand' (1892) see Schirn (1990) and (2000).

uncertain that Frege construed the definition resulting from SUB as one whose *definiens* does not have the same reference as the *definiens* of (ED). Second, the incidental remark fails to furnish conclusive evidence that he toyed with the idea of explicitly defining cardinal numbers as objects other than extensions of concepts. Third, there is likewise no evidence that Frege understood the remark as an allusion to the definability of "$N_x\varphi(x)$" in terms of "the concept *equinumerous with the concept* F". Although it seems that he felt uncomfortable about his introduction of extensions of concepts in *Grundlagen*, this does not mean that he believed he could identify cardinal numbers with objects other than extensions of concepts. Of course, in principle Frege could have defined the number of Fs as the extension of a concept which was not coextensive with the concept *equinumerous with the concept* F. Still, there is no reliable clue that he considered the possibility of "multiple reductions" for the case of cardinal numbers or the other numbers. And, at any rate, the possible identification of the number of Fs with an equivalence class distinct from the class of concepts *equinumerous with the concept* F is not at stake here. The truth is that in the footnote to §68 Frege neither claims nor gainsays that an expression of the form "the concept F" refers to the extension of a concept.

Nevertheless, let me canvass the alternative definition of "$N_x\varphi(x)$", especially with an eye to option (1). By simple virtue of the occurrence of the definite article, its *definiens* is a singular term; it denotes an object. I assume that this is Frege's opinion, though he does not say this expressly; for the requirement that the content (in his theory of definition after 1891: the sense and reference) of the *definiens* be conferred on the *definiendum* can be met only if *definiendum* and *definiens* are expressions belonging to the same syntactic category or logical type. On the face of it, Frege could at least claim that the *definiens* of the alternative definition of "$N_x\varphi(x)$" would be formulated in purely logical terms. According to his view in *Grundlagen* (p. 83), the word "concept" belongs intrinsically to logic. Notice that nowhere in that book does Frege claim that this applies also to "extension of the concept". The problem is, however, that the *definiens* must denote an object, not a concept, if it denotes anything at all. Supposing that the expression "the concept *equinumerous with the concept* F" refers at all to an object, its reference is either (a) the extension of the concept *equinumerous with the concept* F or (b) it is not an extension, but an object of a different kind. If Frege were to choose the first possibility and, hence, were to claim that the two expressions "the concept *equinumerous with the concept F*" and "the extension of the concept *equinumerous with the concept F*" are coreferential, it would be hard to understand why he considers the alternative definition of "$N_x\varphi(x)$" at all. In that case, he would have brought in extensions of concepts anyhow without gaining any advantage from SUB. Firstly, the problem that he simply assumes acquaintance with extensions as logical objects instead of introducing them in a methodologically satisfactory manner would be about the same as the one he is facing in connection with (ED). Secondly, he would have great trouble justifying the claim that "the concept *equinumerous with the concept F*" and "the extension of the concept *equinumerous with the concept F*" refer to the same object. For if the expression "the concept *equinumerous with the concept F*" refers in fact to the extension of the concept *equinumerous with the concept F*, then "the extension of the concept *equinumerous with the concept F*" would denote the extension of an extension.[50]

It is the second possibility (b) that seems to square with the idea Frege insinuates towards the end of *Grundlagen*, namely that extensions of concepts could be dispensed with in pursuit of the logicist programme. It takes little imagination, however, to see that (b) is not only questionable in itself, but also fails to solve the Caesar problem in any plausible sense. Either Frege would have to assume that the objects of kind X, with which cardinal numbers are to be identified via the definition "The number of Fs is the concept *equinumerous with the concept F*" or some other explicit definition of the cardinality operator, are known to us. Or he would have to introduce the Xs in a methodologically sound fashion, presumably by logical abstraction and, thus, by stating a general criterion of identity for them. The assumption that we are familiar with the objects of kind X would probably be even more doubtful as the one which accompanies Frege's formulation of (ED); it would not contribute to solving the Julius Caesar problem. And if he were to introduce the target objects Xs of transsortal identification

[50] Rosado Haddock's objection to this argument that it "is totally unwarranted since extensions do not have extensions" (1998, p. 258) is besides the point. Of course, Frege does not hold that extensions have extensions, but the point of my argument is obviously that he would *nolens volens* be committed to concede that the term "the extension of the concept F" refers to the extension of an extension, if he claimed that "the concept F" refers in fact to the extension of the concept F.

by appeal to a logical abstraction principle, he would have to face a version of the Caesar problem, as is the case when he introduces courses-of-values by means of Axiom V. I, for one, have not even an inkling as to what the objects of kind X might be. So, at the end of the day, the prospects for removing the Caesar problem by explicitly defining cardinal numbers as objects which are not classes appear to be poor in *Grundlagen*. I conclude, then, that option (1) is a rather remote one for Frege, despite his remark that he places no decisive weight on the introduction of extensions of concepts.

(2) In his book *Frege's Conception of Numbers as Objects*, Crispin Wright has suggested a solution of the Caesar problem for the special case of number. According to Wright, the fact that Julius Caesar is distinct from the number of Fs, for any choice of F, follows from two principles. One is HP and the other is a general principle of sortal inclusion. The combination of the two principles leads him to state principle N^d: G(x) is a sortal concept under which numbers fall only if there are, or could be singular terms "a" and "b" purporting to denote instances of G(x) such that the truth-conditions of "a = b" could adequately be explained as those of some statement to the effect that one-one correlation obtains between a pair of concepts (cf. Wright (1983), pp. 116f.). Interesting as Wright's proposal is, it presumably never came to Frege's mind.[51]

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[51] In my view, HP and N^d do not suffice for fixing uniquely the references of numerical singular terms. Although the Zermelo numbers are ruled out from the range of possible candidates for the references of numerical terms, certain sets of Zermelo numbers are still candidates for the objects which could be referred to by numerical singular terms. As a matter of fact, HP and N^d allow an infinitely broad spectrum for the identification of numbers with certain sets or classes. Arguing in the wake of Wright (1983), Hale (1987, p. 206) has suggested strengthening N^d by laying down the following principle S: Singular terms from a given range stand for instances of a sortal concept F if and only if there is some sortal G, whose extension is included in that of F, such that, where "a" and "b" are terms from that range, understanding "a = b" involves exercising a grasp of the criterion of identity for Gs. Hale's answer to the question to which objects the numerical terms in fact refer to (see p. 213) can claim plausibility only if we grant him two assumptions he makes, namely that numbers are classes and that classes are objects. If we drop these assumptions, it could well be that no object falls under the concept of natural number. The principles HP, N^d and S, which are entirely restricted to cardinal aspects and, thus, ignore ordinal aspects, show at best that the concept of natural number is a sortal concept and that therefore all instantiations of this concept are objects. I am indebted here to Robert Bublak.

(3) In my eyes, there is probably only one option to solve the Caesar problem without relying on extensions of concepts, which Frege might have contemplated seriously from the point of view of his overall approach in *Grundlagen*. It is this: to resume (CD) and make a sound additional stipulation. For if (CD) has to go overboard only because it causes the Caesar problem, then, from Frege's viewpoint, it should be possible to reestablish (CD) if this problem can be overcome without (ED) and, hence, without recourse to extensions. As was remarked earlier, the additional stipulation must satisfy a fundamental condition: it must be consistent with HP. Furthermore, it is not allowed to be based on intuition or experience; otherwise it would jeopardize, if not undermine, the assumed logical nature of cardinal numbers. If we keep in mind that Frege regards the identification of the two truth-values with their own unit classes in §10 of *Grundgesetze* as the key for overcoming the referential indeterminacy of course-of-values terms, we could tentatively propose the following "solution" for cardinal numbers or numerical terms: Suppose that the domain of the first-order variables of the formal theory, within which the logicist programme as outlined in *Grundlagen* is to be carried out, contains only cardinal numbers and the two truth-values conceived of as objects. In that case, Frege would be free to stipulate, by invoking a special "permutation argument" and without contradicting (CD), that the True shall be identical with a number belonging to a first-level concept F, and the False with a number belonging to any other first-level concept G which is not equinumerous with F. I do not claim that in writing *Grundlagen* Frege ever considered such a stipulation and I am, of course, aware that at this stage of his working life he had not yet introduced the truth-values as objects. Moreover, I do not wish to pretend that by pursuing this strategy he would have succeeded in removing the problem of the referential indeterminacy of the cardinality operator. On the one hand, the hypothetical identification of the True and the False with certain cardinal numbers does not alter the fact that (CD) fails to fix the truth-value of an equation like "The number belonging to the concept *identical with itself* = the number belonging to the concept *finite cardinal number*". On the other hand, Frege's permutation argument in §10 of *Grundgesetze* seems to show that the referential indeterminacy of course-of-values terms (and this would equally apply to the case of numerical terms) cannot be remedied by means of an additional stipulation (see Dummett (1981), pp. 421 ff.). Nonetheless, I do think that my idea is worth

considering in the present context. My aim was only to point out which option for resolving the Caesar problem Frege might have chosen from *his* point of view, had he dispensed with extensions of concepts, while at the same time adhering to (CD). In saying this, I assume, for the sake of argument, that around 1884 he had already introduced the truth-values into his logical theory and had stipulated that its first-order domain is to comprise only cardinal numbers and the True and the False. We must not lose sight of another important point in this context, though. If Frege had indeed thought he had contrived an additional stipulation which was unassailable and effectively removed the referential indeterminacy of numerical terms of the form "$N_xF(x)$", it would be hard to fathom why he did not present such a solution in §66 or §67 of *Grundlagen*.

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BS *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle a.S. 1879.

GGA *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet*, vol. I, Jena 1893, vol II, Jena 1903.

GLA *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*, Breslau 1884.

KS *Kleine Schriften*, ed. I. Angelelli, Hildesheim 1967.

NS *Nachgelassene Schriften*, eds. H. Hermes, F. Kambartel and F. Kaulbach, Hamburg1969.

WB *Wissenschaftlicher Briefwechsel*, eds. G. Gabriel, H. Hermes, F. Kambartel, C. Thiel, A. Veraart, Hamburg 1976.

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