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PHYSICS, PHILOSOPHY, AND THE FOUNDATIONS OF GEOMETRY

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Twentieth century philosophy of geometry is a development and continuation, of sorts, of late nineteenth century work on the mathematical foundations of geometry. Reichenbach, Schlick, and Carnap appealed to the earlier work of Riemann, Helmholtz, Poincaré, and Hilbert, in particular, in articulating their new view of the nature and character of geometrical knowledge. This view took its starting point from a rejection of Kant's conception of the synthetic a priori status of specifically Euclidean geometry—and, indeed, from a rejection of any role at all for spatial intuition within pure mathematics. We must sharply distinguish, accordingly, between pure or mathematical geometry, which is an essentially uninterpreted axiomatic system making no reference whatsoever to spatial intuition or any other kind of extra-axiomatic or extra-formal content, and applied or physical geometry, which then attempts to coordinate such an uninterpreted formal system with some domain of physical facts given by experience. Since, however, there is always an optional element of decision in setting up such a coordination in the first place (we may coordinate the uninterpreted terms of purely formal geometry with light rays, stretched strings, rigid bodies, or whatever), the question of the geometry of physical space is not a purely empirical one but rather essentially involves an irreducibly conventional and in some sense arbitrary choice. In the end, there is therefore no fact of the matter whether physical space is Euclidean or non-Euclidean, except relative to one or another essentially arbitrary stipulation coordinating the uninterpreted terms of pure mathematical geometry with some or another empirical physical phenomenon.

This characteristically twentieth century view in the philosophy of geometry took its inspiration, as well, from Einstein's creation of the general theory of relativity—which theory was taken by logical empiricist philosophers like Reichenbach, Schlick, and Carnap to be the culmination and epitome, as it were, of nineteenth century research into the mathematical foundations of geometry by such thinkers as Riemann, Helmholtz, Poincaré, and Hilbert.[1] Nevertheless, this close association between the general theory of relativity and the conventionalist view in the philosophy of geometry sketched above can appear extremely puzzling from our present point of view. According to the general theory of relativity, as we now understand it, the geometry of physical space becomes, in effect, another physical field—the field mediating specifically gravitational interactions. Whether a given region of physical space is Euclidean or non-Euclidean depends on the distribution of matter and energy in that region and therefore appears to be a straightforwardly empirical or physical question: geometry, as it is sometimes said, has become a branch of physics.[2] So there appears to be no warrant, from this perspective, for the view that physical geometry is in any way arbitrary or conventional. There is a fact of the matter about the geometry of physical space, and Euclidean geometry,

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[1] The most explicit development of the characteristically twentieth century view I am considering is in Reichenbach's *Philosophie der Raum-Zeit-Lehre* (Berlin: de Gruyter, 1928); translated as *The Philosophy of Space and Time* (New York: Dover, 1957). In particular, the essence of the view is presented in §§ 4 - 8 of this work, beginning with a section on "coordinative definitions" and culminating in a statement of "the relativity of geometry." And the connection with Einstein's theory becomes even more explicit when we note, as Reichenbach himself points out, that the view he is presenting is a logical development of views earlier presented in the context of discussions of the foundations of general relativity theory by Schlick and himself. Carnap, in his Introductory Remarks to the English Edition, briefly sketches the view in question and explains that "[t]he view just outlined concerning the nature of geometry in physics stresses, on the one hand, the empirical character of physical geometry and, on the other hand, recognizes the important function of conventions[; t]his view was developed in the twenties of our century by those philosophers who studied the logical and methodological problems connected with the theory of relativity, among them Schlick, Reichenbach, and myself" (p. vi). For discussion of the earlier views of Schlick, Reichenbach, and Carnap on the foundations of geometry and relativity theory see my *Reconsidering Logical Positivism* (Cambridge: Cambridge University Press, 1999), Part One.

[2] This is the title of a well-known paper by the physicist H. P. Robertson, published in P. Schilpp, ed., *Albert Einstein: Philosopher-Scientist* (La Salle: Open Court, 1949), pp. 315-32.

in particular, is, in a perfectly straightforward sense, simply a false description of the overall structure of this space (although it may hold, in the limit, in certain precisely defined physical situations).

The *locus classicus* for the intersection between twentieth century philosophy of geometry and Einstein's general theory of relativity is a famous paper by Einstein himself, entitled "Geometry and Experience" and delivered in 1921.[3] This paper articulates a very clear and sharp version of the distinction between "pure" and "applied"—mathematical and physical—geometry that soon became canonical in twentieth century scientific thought.[4] According to Einstein, mathematical geometry derives its certainty and purity from its "formal-logical" character as a mere deductive system operating with "contentless conceptual schemata." The primitive terms of mathematical geometry, such as "point," "line," "congruence," and so on, do not refer to objects or concepts antecedently given (by some sort of direct intuition, for example) but rather have only that purely "formal-logical" meaning stipulated in the primitive axioms. These axioms serve therefore as "implicit definitions" of the primitive terms, and all the theorems of mathematical geometry then follow purely logically from the stipulated axioms.[5] Applied or physical geometry, by contrast, arises when one gives some definite extra-axiomatic interpretation of the primitive terms via real objects of experience. But now the purity and certainty of

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[3] A. Einstein, "Geometrie und Erfahrung," *Preussische Akademie der Wissenschaft. Physikalischmathematische Klasse. Sitzungsberichte* (1921), pp. 123-30; *Erweiterte Fassung des Festvortrages gehalten an der Preussischen Akademie der Wissenschaft zu Berlin am 27. Januar 1921* (Berlin: Springer, 1921); translated as "Geometry and Experience," in G. Jeffrey and W. Perrett, eds., *Sidelights on Relativity* (London: Methuen, 1923), pp. 27-55. In what follows I present material originally presented in my "Geometry as a Branch of Physics: Background and Context for Einstein's 'Geometry and Experience'," in D. Malament, ed., *Reading Natural Philosophy* (Chicago: Open Court, 2002). I am indebted to David Malament and Open Court Publishing Company for permission to present this material here.

[4] A well-known later presentation of this canonical twentieth century view is C. Hempel, "Geometry and Empirical Science," *American Mathematical Monthly* 52 (1945); reprinted in H. Feigl and W. Sellars, eds., *Readings in Philosophical Analysis* (New York: Appleton-Century-Crofts, 1949), pp. 238-49. Hempel's paper is basically an elementary exposition of the first part of Einstein's; it closes by quoting Einstein's famous characterization of the relationship between mathematical certainty and empirical reality.

[5] See *Geometrie und Erfahrung. Erweiterte Fassung* (note 3), pp. 4-5 (Jeffrey and Perrett, pp. 30-31).

mathematical geometry (which, in the end, rests simply on the purity and certainty of logic) is irrevocably lost, and we end up with one more *empirical* science among others:

> In so far as the propositions of mathematics refer to reality they are not certain; and in so far as they are certain they do not refer to reality. Full clarity about the situation appears to me be have been first obtained in general by that tendency in mathematics known under the name of "axiomatics." The advance achieved by axiomatics consists in having cleanly separated the formal-logical element from the material or intuitive content. According to axiomatics only the formal-logical element constitutes the object of mathematics, but not the intuitive or other content connected with the formal-logical element.[6]

Thus, these famous words from Einstein's paper, which were clearly intended and were standardly taken as a refutation of the Kantian conception that mathematics is the paradigm of synthetic a priori truth, are a vivid expression of the modern axiomatic conception of geometry we now associate with the work of Hilbert. Indeed, Einstein himself takes the notion of "implicit definition" to which he appeals here from Schlick's generalization of Hilbert's point of view to all of empirical science in his 1918 treatise on *General Theory of Knowledge*.[7]

So far, then, Einstein's own view of the matter is perfectly in harmony with the characteristically twentieth century view in the philosophy of geometry with which I began. In articulating his view of specifically physical geometry, however, Einstein takes a quite different tack. For he here views physical geometry, in particular, as a straightforward empirical theory of the actual physical behavior of "practically rigid bodies," and he claims, in a striking passage, that "without [this conception] I would have found it impossible to establish the [general] theory of relativity." Immediately thereafter, in the same passage, Einstein considers Poincaré's geometrical conventionalism—apparently as the only real alternative to his own view—and suggests that "if one [following Poincaré] rejects the relation between the practically

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[6] *Geometrie und Erfahrung*, pp. 3-4 (Jeffrey and Perrett, pp. 28-9). (All translations from German originals are my own.)

[7] M. Schlick, *Allgemeine Erkenntnislehre* (Berlin: Springer, 1918); translated (from the second, 1925 edition) as *General Theory of Knowledge* (La Salle: Open Court, 1985). This is one of the earlier works to which Reichenbach refers in *The Philosophy of Space and Time* (note 1).

rigid body and geometry one will in fact not easily free oneself from the convention according to which Euclidean geometry is to be held fast as the simplest." Einstein concedes that *"[s]ub specie aeterni* Poincaré, in my opinion is correct," for "practically rigid bodies" are in fact unsuitable to play the role of "irreducible elements in the conceptual framework of physics." Nevertheless, Einstein suggests, they must *provisionally* "still be called upon as independent elements in the present stage of theoretical physics"—when, in particular, we are still very far from an adequate micro-theory of the structure of matter. And where such rigid bodies must "still be called upon as independent elements," it is clear, is precisely in the foundations of the general theory of relativity.[8]

In thus considering applied or physical geometry as a straightforward empirical theory, Einstein appears to be much more in harmony with our present, post-conventionalist understanding of general relativity. From the point of view of precisely this current understanding, however, there are other aspects of Einstein's overall view that now appear puzzling in a rather different way. Why, in the first place, should a sharp distinction between pure or axiomatic and applied or physical geometry be specifically associated with the theory of relativity? Einstein is well-aware, as I suggested, that this distinction had most recently become prominent in connection with Hilbert's celebrated axiomatization of Euclidean geometry in 1899—a development that occurred some fifteen years before Einstein's own application of non-Euclidean geometry in physics.[9] So the distinction has, on the face of it, no special relevance to this later development, and it makes just as much (or as little) sense in the context of the traditional use of Euclidean geometry in classical physics. Indeed, it makes just as much (or as little) sense in the context of everyday applications of Euclidean geometry in surveying and measurement—which Einstein, in "Geometry and Experience," calls "the oldest branch of physics"[10] —as it does in the context of the very complex and sophisticated application of non-Euclidean geometry employed by Einstein's new theory. As a matter of

[8] See *Geometrie und Erfahrung,* pp. 5-8 (Jeffrey and Perrett, pp. 31-6).

[9] D. Hilbert, *Grundlagen der Geometrie* (Leipzig: Teubner, 1899); translated (from the tenth, 1968 edition) as *Foundations of Geometry* (La Salle: Open Court, 1971). This work is of course prominently cited by Schlick as the basis for the notion of "implicit definition" in his *General Theory of Knowledge* (note 7), § 7.

[10] See *Geometrie und Erfahrung,* pp. 5-6 (Jeffrey and Perrett, pp. 31-3).

fact, and in the second place, Einstein's particular explanation of applied or physical geometry actually appears to be *more* appropriate to everyday applications in surveying and measurement that it does to the general theory of relativity. For it is not as if, in the general theory, we find that space is non-Euclidean by actually performing surveying experiments with rigid measuring rods. Rather, we postulate a highly theoretical link between gravitation and spatio-temporal curvature, on the basis of which it then emerges from some very abstract mathematics that the spatial region in the neighborhood of the sun, for example, is described by a non-Euclidean geometry. This conclusion can then be tested, of course, but the whole procedure bears very little resemblance to the surveyor's conception of applied geometry Einstein articulates in "Geometry and Experience."

In order to begin making sense of Einstein's argument, we need to consider it against the background of a preceding conception of geometry—one that was dominant in the late nineteenth century but has now, largely through the influence of "modern" views like Einstein's, receded far into the background in contemporary philosophical discussion. This earlier tradition had its home in projective geometry and group theory, and it found its canonical expression in the famous Erlanger program of Felix Klein.[11] Here, as Klein himself puts it, pure or mathematical geometry is by no means conceived as an "empty conceptual schema," but is rather understood as necessarily connected to our "spatial intuition."[12] Mathematical geometry, on this view,

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[11] For discussion of Klein's Erlanger program see R. Torretti, *Philosophy of Geometry from Riemann to Poincaré* (Dordrecht: Reidel, 1978), § 2.3. Torretti's *Relativity and Geometry* (Oxford: Pergamon, 1983) depicts the development of relativity theory as the outcome of two quite different nineteenth century programs for unifying the foundations of geometry—Klein's Erlanger program, on the one side, and Riemann's general theory of manifolds, on the other. The special theory of relativity can be seen as based on the former program, the general theory on the latter. As Torretti masterfully shows, however, serious misconceptions arise by not keeping these different traditions distinct; for, as further explained below, the former applies only to spaces of constant curvature and can therefore be of no use whatsoever in characterizing the variably curved space-time structure of general relativity. The present paper, in this respect, can be thought of as a further development and application of Torretti's original insight.

[12] Klein's *Elementarmathematik vom höheren Standpunkt aus. Teil II: Geometrie* (Leipzig: Teubner, 1909), pp. 383-4–translated as *Elementary Mathematics from an Advanced Standpoint: Geometry* (New York: Dover, 1939), p. 187—puts the matter as follows: "In the case of such people who are only interested in the logical

describes the most general and abstract features of our perception of space—namely, the "perspectival" features of space as we move in and through it and perceive spatial objects from different point of view. These features are not precise and specific enough to yield Euclidean geometry in particular, however, but only that structure common to the three classical geometries of constant curvature (Euclidean, hyperbolic, and elliptic), which, in terms of the Erlanger program, emerge naturally within the more general framework of projective geometry and group theory. So only considerations of convenience and expediency (especially simplicity), not deliverances of our spatial intuition, can then explain our choice of specifically Euclidean geometry. This view of geometry has obvious roots in the conception articulated and defended by Kant at the end of the eighteenth century, but it aims to *generalize* the Kantian picture to take account of nineteenth century discoveries in projective and non-Euclidean geometry.[13]

side of the question, and not in the intuitive or general-epistemological side, one often finds the opinion nowadays that *the axioms are only arbitrary propositions that we set up entirely freely, and the fundamental concepts, ultimately, are also only arbitrary signs for things with which we wish to operate*. What is correct in such a view, of course, is that *within pure logic* no basis for these propositions and concepts is found, and that they therefore must be furnished or suggested from another side—precisely by the influence of intuition. However, the authors [in question] often express themselves much more one-sidedly, and so we are repeatedly forced nowadays, in connection with modern axiomatics, straightaway once again into that philosophical position which has been known since ancient times as *nominalism*. Here the interest in the things themselves and their properties is entirely lost; and one speaks only of how they are to be named and in accordance with which logical schema they are to be operated. One then says, for example, that we *call* a triple of coordinates a point 'without thereby thinking of anything,' and we stipulate 'arbitrarily' certain propositions that are to be valid of these points; one can set up arbitrary axioms in an entirely unlimited way, so long as one satisfies the laws of logic and takes care, above all, that there is no contradiction in the ensuing structure of theorems. I myself in no way share this standpoint but take it to be the death of all science: *the axioms of geometry*—in my opinion—*are not arbitrary but rather* **rational** *propositions, which are motivated, in general, by spatial intuition and regulated, in their particular content, by considerations of purposiveness* [*Zweckmäßigkeitsgründe*]." The context makes it clear that Klein has those excessively influenced by Hilbertian axiomatics in mind. Hilbert's own work in the foundations of geometry grew out of this same projective tradition, however, and Hilbert himself famously says in the brief Introduction to his *Foundations of Geometry* (note 9) that the problem of axiomatizing geometry amounts to "the logical analysis of our spatial intuition."

[13] See Torretti, *Philosophy of Geometry* (note 11), § 2.3.10 for a discussion of Klein's view of spatial intuition. For a somewhat more sympathetic discussion of the

For our present purposes, the most important result of this tradition is what we now call the Helmholtz-Lie theorem, which was first articulated by Helmholtz in connection with his psycho-physiological researches into space perception and then rigorously proved by Sophus Lie (at the instigation of his teacher Klein) within Lie's theory of continuous groups. Helmholtz was inspired by Riemann's work on "n-fold extended manifolds" to attempt to derive Riemann's fundamental assumption, that the line-element is Pythagorean or infinitesimally Euclidean, from what Helmholtz took to be the fundamental "facts" generating our perceptual intuition of space.[14] Helmholtz's starting point was that our idea of space is in no way immediately given or "innate" but instead arises by a process of perceptual accommodation or learning based on our experience of bodily motion. Since our idea of space arises kinematically, as it were, from our experience of moving up to, away from, and around the objects that "occupy" space, the space thereby constructed must satisfy a condition of "free mobility" that permits arbitrary continuous motions of rigid bodies.[15] And from this latter condition one can then derive the Pythagorean form of the line-

relationship between (closely related) nineteenth centuries views of spatial intuition and the original Kantian conception see my "Geometry, Construction, and Intuition in Kant and His Successors," in G. Scher and R. Tieszen, eds., *Between Logic and Intuition: Essays in Honor of Charles Parsons* (Cambridge: Cambridge University Press, 2000).

[14] B. Riemann, "Über die Hypothesen, welche der Geometrie zugrunde liegen," *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 13 (1867); *Neu herausgegeben und erläutert von H. Weyl* (Berlin: Springer, 1919); translated as "On the Hypotheses which Lie at the Foundations of Geometry," in D. Smith, ed., *A Source Book in Mathematics,* vol. 2 (New York: Dover, 1959), pp. 411-25. H. Helmholtz, "Über die Tatsachen, die der Geometrie zum Grunde liegen," *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-August-Universität aus dem Jahre 1868* 9 (1868); reprinted in H. Helmholtz, *Wissenschaftliche Abhandlungen,* vol. 2 (Leipzig: Barth, 1883); translated as "On the Facts Underlying Geometry," in R. Cohen and Y. Elkana, eds., *Hermann von Helmholtz: Epistemological Writings* (Dordrecht: Reidel, 1977), pp. 39-71.

[15] For Helmholtz's view of space-perception see G. Hatfield, *The Natural and the Normative: Theories of Space Perception from Kant to Helmholtz* (Cambridge, Mass.: MIT Press, 1990), chapter 5. For a discussion of Helmholtz's mathematical results in the context of his theory of space-perception see J. Richards, "The Evolution of Empiricism: Hermann von Helmholtz and the Foundations of Geometry," *British Journal for the Philosophy of Science* 28 (1977), pp. 235-53. See also my "Helmholtz's *Zeichentheorie* and Schlick's *Allgemeine Erkenntnislehre:* Early Logical Empiricism and Its Nineteenth-Century Background," *Philosophical Topics* 25 (1997), pp. 19-50.

element.[16] Since, however, the Riemannian metric thereby constructed has a group of isometries or rigid motions mapping any point onto any other, it must have constant curvature as well. So the scope of the Helmholtz-Lie theorem (and the entire Kleinian tradition) is much less general than the full Riemannian theory of metrical manifolds, which of course also includes manifolds of *variable* curvature.

The Helmholtz-Lie theorem fixes the geometry of space—and, according to Helmholtz, thereby expresses the "necessary form of our outer intuition"—as one of the three classical geometries of constant curvature: Euclidean, hyperbolic, or elliptic. But how do we know which of the three classical geometries actually holds? At this point, on Helmholtz's view, we investigate the actual behavior of rigid bodies (of rigid measuring rods, for example) as we move them around in accordance with the condition of free mobility. That physical space is Euclidean (which Helmholtz of course assumed) means that physical measurements carried out in this way are empirically found to satisfy the laws of this particular geometry to a very high degree of exactness. Thus Helmholtz's view is Kantian in so far as space indeed has a "necessary form" expressed in the condition of free mobility, but it is empiricist in so far as which of the three possible geometries of constant curvature actually holds is then determined by experience.[17]

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16 In Lie's formulation, given a group of transformations on a manifold such that, intuitively, for any two "observers" or "points of view" there is exactly one transformation in the group mapping one onto the other, there is a unique—up to a scale factor—Riemannian metric on the manifold whose isometries are given precisely by the group in question. For the work of Helmholtz and Lie see Torretti, *Philosophy of Geometry*, § 3.1. For a philosophically and mathematically sophisticated discussion of Helmholtz and Riemann see §§ VI-VII of H. Stein, "Some Philosophical Prehistory of General Relativity," in J. Earman, C. Glymour, and J. Stachel, eds., *Minnesota Studies in the Philosophy of Science*, vol. VIII (Minneapolis: University of Minnesota Press, 1977), pp. 3-49; footnote 29, in particular, presents an up-to-date exposition of the mathematics of the Helmholtz-Lie theorem.

17 Helmholtz characterizes space as a "subjective *form* of intuition" in the sense of Kant, and as the "*necessary* form of our outer intuition," in his famous address on "Die Tatsachen in der Wahrnehmung" of 1878. See P. Hertz and M. Schlick, eds., *Hermann v. Helmholtz: Schriften zur Erkenntnistheorie* (Berlin: Springer, 1921), p. 117; translated as "The Facts in Perception," in Cohen and Elkana, eds. (note 14), p. 124. Helmholtz viewed the condition of free mobility, in particular, as a necessary condition of the possibility of spatial measurement, and thus of the application of geometry. For discussion see the works cited in notes 15 and 16 above.

Now it was precisely this Helmholtzian view of physical geometry that set the stage, in turn, for the contrasting "conventionalist" conception articulated by Poincaré. Indeed, Poincaré developed his philosophical conception immediately against the background of the Helmholtz-Lie theorem, and in the context of his own mathematical work on group theory and models of hyperbolic geometry.[18] Following Helmholtz and Lie, Poincaré viewed geometry as the abstract study of the group of motions associated with our initially crude experience of bodily "displacements." So we know, according to the Helmholtz-Lie theorem, that the space thereby constructed has one and only one of the three classical geometries of constant curvature. Poincaré disagreed with Helmholtz, however, that we can empirically determine the particular geometry of space simply by observing the behavior of rigid bodies. No real physical bodies exactly satisfy the condition of geometrical rigidity, and, what is more important, knowledge of physical rigidity presupposes knowledge of the forces acting on the material constitution of bodies. But how can one say anything about such forces without first having a geometry in place in which to describe them? We have no option, therefore, but to *stipulate* one of the three classical geometries of constant curvature, by convention, as a framework within which we can then do empirical physics.[19] Moreover, since Euclidean geometry is mathematically the simplest, Poincaré had no doubt at all that this particular stipulation would always be preferred.

We know that Einstein was intensively reading Poincaré when he was creating the special theory of relativity in 1905, and it seems very plausible, accordingly, that Poincaré's conventionalism played a significant role in philosophically motivating this theory.[20] More specifically, whereas Poincaré had argued, against both Kant and Helmholtz, that the particular geometry of space is not dictated by

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[18] Poincaré discovered his well-known models of hyperbolic geometry in the context of his work on "Kleinian groups" in complex analysis—which he then found, surprisingly, to include the isometries of hyperbolic geometry. Poincaré describes this famous discovery in the chapter on "Mathematical Invention" in *Science et Méthode* (Paris: Flammarion, 1908); translated as *Science and Method,* in G. Halsted, ed., *The Foundations of Science* (Lancaster: The Science Press, 1913). For a discussion of the Poincaré models see Torretti, *Philosophy of Geometry,* § 2.3.7.

[19] For a detailed analysis of Poincaré's geometrical conventionalism along these lines see my *Reconsidering Logical Positivism* (note 1), chapter 4.

[20] See A. Miller, *Albert Einstein's Special Theory of Relativity* (Reading: Addison-Wesley, 1981), chapter 2.

either reason or experience but rather requires a fundamental decision or convention of our own, Einstein now argues, similarly, that simultaneity between distant events is not dictated by either reason or experience but requires a new fundamental *definition* based on the behavior of light.[21] Moreover, Einstein proceeds here, in perfect conformity with Poincaré's underlying philosophy, by "elevating" an already established empirical fact—the invariant character of the velocity of light in different inertial reference frames—into the radically new status of what Poincaré calls a convention or "definition in disguise" (here a definition of simultaneity).[22]

As we also know, however, Einstein tells us in "Geometry and Experience" that he needed to reject Poincaré's geometrical conventionalism in order to arrive at the *general* theory of relativity. In particular, Einstein here adopts a Helmholtzian conception of (applied or physical) geometry as a straightforward empirical theory of the actual physical behavior of "practically rigid bodies," and he claims that "without [this conception] I would have found it impossible to establish the [general] theory of relativity." Here, as Einstein explains in the same passage, he has in mind the following line of thought.[23] According to the principle of equivalence (based on the equality of gravitational and inertial mass) gravitation and inertia are essentially the same phenomenon. So, in particular, we can model gravitational fields by "inertial fields" (involving centrifugal and Coriolis forces, for example) arising in non-inertial frames of reference. If we now consider a uniformly rotating frame of reference in the context of special relativity, we then find that the Lorentz contraction differentially affects measuring rods laid off along concentric circles around the origin in the plane of rotation (due to the variation in tangential linear velocity at different distances along a radius), whereas no Lorentz contraction is experienced by rods laid off along a radius. Therefore, the geometry in a rotating

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[21] Indeed, Poincaré had himself already argued that distant simultaneity requires a convention or definition—also involving the velocity of light—in "La measure du temps," *Revue de Métaphsique et de Morale* 6 (1898), pp. 1-13; translated as "The Measure of Time," in Halsted ed. (note 18), pp. 223-34.

[22] For further discussion of this idea of "elevating" an already established empirical fact to the status of what Poincaré calls a convention—with special reference to the example of special relativity and the velocity of light—see my *Dynamics of Reason* (Stanford: CSLI, 2001), pp. 86-9.

[23] See *Geometrie und Erfahrung*, pp. 6-7 (Jeffrey and Perrett, pp. 33-4).

system will be found to be non-Euclidean (the ratio of the circumference to the diameter of concentric circles around the origin in the plane of rotation will differ from π and depend on the circular radius).

The importance of this line of thought for Einstein is evident in virtually all of his expositions of the general theory, where it is always used as the primary motivation for introducing non-Euclidean geometry into the theory of gravitation.[24] Moreover, as John Stachel has shown, this particular thought experiment in fact constituted the crucial breakthrough to what we now know as the mathematical and conceptual framework of general relativity.[25] For, generalizing from this example, Einstein quickly saw that what he really needed for a relativistic theory of gravitation is a *four dimensional* version of non-Euclidean geometry (comprising both space and time). He quickly saw that a variably curved generalization of the flat Minkowski metric of special relativity should serve as the representative of the gravitational field, and, turning to the mathematician Marcel Grossmann for help, he then discovered the Riemannian theory of manifolds. Einstein's repeated appeal to the example of the uniformly rotating frame of reference in his official expositions of the theory therefore appears to reflect the actual historical process of discovery very accurately, and to explain, in particular, how the idea of a variably curved four dimensional space-time geometry was actually discovered in the first place.[26]

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[24] See, for example, "Die Grundlage der allgemeinen Relativitätstheorie," *Annalen der Physik* 49 (1916), p. 776—translated as "The Foundation of the General Theory of Relativity," in H. Lorentz, et. al., *The Principle of Relativity* (London: Methuen, 1923), p. 117; *Über die spezielle und die allgemeine Relativitätstheorie, gemeinverständlich* (Braunschweig: Vieweg, 1917), §§ 23-8—translated (from the fifth, 1918 edition) as *Relativity, The Special and the General Theory: A Popular Exposition* (London: Methuen, 1920); *The Meaning of Relativity* (Princeton: Princeton University Press, 1922), pp. 59-61.

[25] See J. Stachel, "Einstein and the Rigidly Rotating Disk," in A. Held, ed., *General Relativity and Gravitation* (New York: 1980); reprinted as "The Rigidly Rotating Disk as the 'Missing Link' in the History of General Relativity," in D. Howard and J. Stachel, eds., *Einstein and the History of General Relativity* (Boston: Birkhäuser, 1989), pp. 48-62.

[26] As is well known, Einstein (and Grossmann) required several more years of struggle to find the final field equations of general relativity—during which, in particular, they were sidetracked by the now notorious "hole argument." See J. Stachel, "Einstein's Search for General Covariance, 1912-1915," (first presented in 1980) in Howard and Stachel, eds. (note 25), pp. 63-100; J. Norton, "How Einstein

Einstein's introduction of non-Euclidean geometry into physics thus followed a remarkably circuitous route and involved, in particular, a process by which Einstein delicately positioned himself within the debate on the foundations of geometry between Helmholtz and Poincaré—a debate which was itself framed, as we have seen, by a nineteenth century tradition of mathematical work in group theory and projective geometry. In creating the special theory of relativity in 1905 Einstein took inspiration from Poincaré's conventionalist scientific epistemology—not, however, as applied to spatial geometry but rather to what we now call the (four dimensional) geometry of space-time. Once the special theory was in place Einstein then faced a radically new situation in the theory of gravitation, for the Newtonian theory of universal gravitation—based, as it was, on an instantaneous action at a distance and thus on absolute simultaneity—was itself incompatible with the new conceptual structure of the special theory based on a relativized conception of simultaneity. Einstein was therefore faced with the problem of adjusting the theory of gravitation to this new relativistic conceptual structure, and he addressed this problem, in the first instance, by appealing to the already well-established empirical fact that gravitational and inertial mass are equal. This fact led him to his principle of equivalence—the idea that gravitation and inertia are the very same physical phenomenon—which he then applied, as we have seen, to non-inertial frames of reference (accelerating and rotating frames) within the conceptual structure of the special theory (within what we now call Minkowski space-time).[27] This led him, in turn, by the example of the uniformly rotating frame, to a non-Euclidean spatial geometry (now linked to the action of a gravitational field), which he was then able, finally, to generalize to the non-Euclidean *space-time* geometry of general relativity.

Moreover, the crucial thought-experiment of the uniformly rotating frame of reference essentially involved, as Einstein tells us in "Geometry

Found His Field Equations, 1912-1915," *Historical Studies in The Physical Sciences* 14 (1984), reprinted in Howard and Stachel, eds., pp. 101-59. The present point, however, is that the essential mathematical structure required by general relativity—the idea of representing gravitation by a variably curved four dimensional space-time metric—had already been articulated by 1912.

27 For an outstanding analysis of Einstein's use of the principle of equivalence in this connection see J. Norton, "What Was Einstein's Principle of Equivalence?" *Studies in History and Philosophy of Science* 16 (1985); reprinted in Howard and Stachel, eds., pp. 5-47.

and Experience," a naively Helmholtzian rather than a sophisticated Poincaré-inspired perceptive on the relationship between the behavior of rigid bodies and physical geometry. Indeed, it was necessary for Einstein to have already rejected the more sophisticated perspective on rigid bodies suggested by Poincaré in creating the special theory of relativity. For, as is well known, Poincaré was actually the first person to discover what we now know as the Lorentz group governing inertial reference frames in special relativity, and Poincaré had formulated, accordingly, a Lorentzian version of the mathematics of special relativity still in some sense committed to a classical aether.[28] For Poincaré, we might say, the Lorentz group thus operated at the level of electrodynamics—governing the microscopic electromagnetic structure ultimately responsible for physical rigidity—but not, as in Einstein, at the more fundamental kinematical level governing the basic concepts of space, time, and motion formulated prior to and independently of any particular dynamical theory. Just as, in the special theory, Einstein takes the Lorentz contraction as a direct indication of fundamental *kinematical* structure, independently of all dynamical questions about the micro-physical forces actually responsible for physical rigidity, here, in the example of the uniformly rotating reference frame, Einstein similarly takes the Lorentz contraction as a direct indication of fundamental *geometrical* structure. And without this remarkably circuitous procedure of delicately situating himself, as it were, between Helmholtz and Poincaré, it is indeed hard to imagine how Einstein could have ever discovered the idea of a variably curved four dimensional space-time geometry in the first place.[29]

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[28] See Miller, *Einstein's Special Theory* (note 20), § 1.14. In the Lorentz-Fitzgerald theory of the electron—a version of which Poincaré accepts—the Lorentz contraction is of course a genuine physical or dynamical phenomenon (judged from the point of the privileged "aether" frame), ultimately due to the microphysical effects of electromagnetic forces.

[29] From our present, post-general-relativistic point of view, the most natural procedure is to begin with the flat four dimensional geometry of Minkowski space-time and then use the principle of equivalence to motivate the idea that freely falling trajectories can be conceived as geodesics in a variably curved "perturbation" of an initially flat Minkowski geometry. This line of thought was definitely not available in the actual historical context within which general relativity was created, however. For, on the one hand, Einstein did not appreciate the importance of Minkowski's four dimensional reformulation of special relativity until after he had already created the general theory. And, on the other hand, no one but Einstein had the idea of exploiting the well-known equivalence between gravitational and inertial mass. In

There is an important sense, however, it which this same idea of a variably curved space-time geometry, once discovered, renders the preceding debate between Helmholtz and Poincaré quite irrelevant. For, as we have suggested, this debate is itself framed by the Helmholtz-Lie theorem and is therefore limited—along with the entire Kleinian tradition in group theory and projective geometry within which it is articulated—to spaces of constant curvature. But the general theory of relativity, of course, employs a space-time of variable curvature—dependent on the distribution of matter and energy—where Helmholtz's principle of free mobility therefore fails. As a result, the characteristically late nineteenth century conception of pure geometry as describing the "perspectival" features of our spatial intuition also fails, and we are left with the characteristically twentieth century conception, originally derived from the work of Hilbert, of pure mathematical geometry as an abstract deductive system having no intrinsic relation at all to our spatial perception or any other kind of experience.[30] So it is no wonder, in the end, that Einstein appeals in "Geometry and Experience" to precisely this Hilbertian conception.

Indeed, as we have seen, Einstein bases this appeal, more generally, on Schlick's elaboration of the notion of "implicit definition" in his *General Theory of Knowledge*—which is itself based, in the present context, on Schlick's virtually simultaneous work on the philosophical

particular, Minkowski himself was engaged in an attempt to formulate a relativistic theory of gravitation by a generalized action-at-a-distance theory: for discussion see L. Corry, "Hermann Minkowski and the Postulate of Relativity," *Archive for the History of the Exact Sciences* 51 (1997), pp. 286-92. Thus, the only effective line of thought available at the time was the one Einstein actually followed: beginning from a three dimensional formulation of special relativity we apply the principle of equivalence to (three dimensional) non-inertial reference frames, and we are then led via the example of the rotating frame of reference to a (three dimensional) non-Euclidean spatial geometry—which we are only at this point in a position to generalize to a (four dimensional) non-Euclidean space-time geometry.

[30] Riemann's theory can be seen as an intermediate stage in this development, where geometry in the sense of the theory of manifolds is characterized non-intuitively and in this sense purely conceptually within analysis (in terms of n-tuples of real or complex numbers). It is Hilbert, however, who creates the first purely conceptual (abstract) *synthetic* geometry, and who then clarifies the relationship between geometry in this (axiomatic) sense and analysis via a representation theorem. For discussion of Hilbert's achievement see Torretti, *Philosophy of Geometry*, § 3.2.8.

significance of the general theory of relativity.[31] Here, in particular, Schlick portrays the variably curved space-time of general relativity as an entirely abstract, entirely non-intuitive "conceptual construction," which can only be related to experience and the physical world by an entirely abstract, entirely nonintuitive relation of "designation" or "coordination" in virtue of which the purely mathematical "conceptual construction" represented by Einstein's formulation of the general theory of relativity can then receive empirical content by being interpreted in terms of physical measurement.[32] And it is precisely here, in this procedure of physical coordination, that Schlick finds that Poincaré's conventionalist philosophy of geometry still holds. In particular, we still have a choice whether to use a non-Euclidean or a Euclidean physical geometry even within the context of Einstein's new theory: it is just that in the latter case we would have to introduce further complications into the simple and "natural" physical coordination effected by Einstein's principle of equivalence (which directly coordinates freely falling bodies affected only by gravitation to the four dimensional space-time geodesics of a variably curved, non-Euclidean space-time geometry), and we would thereby introduce onerous complications into our total system of geometry plus physics. But Poincaré had himself maintained that only mathematical simplicity explains our preference for Euclidean *spatial* geometry, and all we are now doing, in the context of Einstein's new theory, is extending Poincaré's viewpoint to *space-time* geometry.[33]

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[31] See M. Schlick, *Raum und Zeit in der gegenwärtigen Physik* (Berlin: Springer, 1917); translated (from the fourth, 1922 edition) as *Space and Time in Contemporary Physics,* in H. Mulder and B. van de Velde-Schlick, eds., *Moritz Schlick: Philosophical Papers,* vol. 1 (Dordrecht: Reidel, 1978), pp. 207-69. Einstein himself had given Schlick considerable help on this very influential exposition of the general theory. For discussion see D. Howard, "Realism and Conventionalism in Einstein's Philosophy of Science: The Einstein-Schlick Correspondence," *Philosophia Naturalis* 21 (1984), pp. 616-29.

[32] See the final chapter of *Space and Time* (note 31), "Relations to Philosophy," which, in turn, is closely related to Schlick's discussion of the "method of coincidences" in *General Theory of Knowledge,* § 30 (§ 31 of the second edition). For further discussion see my "Geometry as a Branch of Physics" (note 3).

[33] Schlick becomes fully clear about this extension of Poincaré's conventionalism to general relativity itself only in the fourth (1922) edition of *Space and Time* and the second (1925) edition of *General Theory of Knowledge.* For discussion see again my "Geometry as a Branch of Physics." The key idea is that Euclidean geometry can be retained by introducing what Reichenbach calls "universal forces" in §§ 5-7 of *The Philosophy of Space and Time.* And we can

This attempt to link Poincaré's conventionalist philosophy of geometry and the general theory of relativity is certainly plausible, and, as we know, it has been extraordinarily influential in twentieth century scientific thought. Nevertheless, it is subject to very deep difficulties from our present, post-conventionalist point of view. In the first place, the problem of coordination as Schlick understands it—the problem of relating an abstract "conceptual construction" to concrete empirical reality—did not exist for Poincaré. Poincaré's own work on geometry—both mathematical and philosophical—is entirely framed, as we have seen, within the late nineteenth tradition in group theory and projective geometry associated with Klein's Erlanger program. Pure geometry, in this tradition, is by no means an uninterpreted axiomatic system but rather an expression of the abstract "perspectival" features of our spatial intuition or perception. Geometry, on this conception, therefore has *space* as its object—the very space in which we live, move, and perceive; and so the problem of coordination or "designation" as Schlick understands simply does not arise. Indeed, this problem, as we have seen, is in an important sense a product of Schlick's assimilation of the radically new conception of space and geometry embodied in Einstein's theory of relativity, not something that was present and available all along.

In the second place, and even more importantly, Poincaré's conception of space and geometry is also entirely based, in accordance with this very same late nineteenth century tradition, on the principle of free mobility first formulated by Helmholtz and later brought to precise mathematical fruition in the Helmholtz–Lie theorem. For it was this principle that formed the indispensable link between pure or

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understand this most clearly and precisely, from our present point of view, in the context of the Cartan-Trautmann reformulation of the original Newtonian theory of gravitation—which is based, like general relativity, on the principle of equivalence and which, accordingly, employs a variably curved four dimensional space-time structure (an affine connection) to represent the action of gravity. When we then recover the traditional formulation—which is based, from a modern point of view, on a flat four dimensional space-time structure (affine connection)—gravity in the traditional (Newtonian) sense appears as precisely a "universal force" in the sense of Reichenbach. Following Reichenbach's methodological prescription to "set universal forces equal to zero," in this context, therefore amounts to adopting the principle of equivalence and rejecting the traditional flat space-time structure. For discussion of the Cartan-Trautmann formulation in this regard see my *Foundations of Space-Time Theories: Relativistic Physics and the Philosophy of Science* (Princeton: Princeton University Press, 1983), §§ III. 4, III.8.

mathematical and applied or physical geometry within the nineteenth century tradition in question. Here the relationship between pure and applied geometry is not of course understood as that between an uninterpreted axiomatic system and a possible (empirical) interpretation of that system; it is rather understood as a relationship between the otherwise empty space in which material bodies are contained and these material bodies themselves—as a relationship, in the original Kantian sense, between a form of intuition and the physical objects or material "content" contained within that form. Unlike in the original Kantian conception, however, we are now operating within a generalization of the notion of form of intuition to include all spaces of constant curvature; and, in this context, the principle of free mobility then serves as our crucial coordinating principle—our crucial link between pure and applied geometry—by coordinating purely geometrical notions, like that of geometrical equality or congruence, for example, to the idealized behavior of physical rigid bodies.[34]

And it is precisely here, in the context of the Helmholtz-Lie theorem, that a remarkable conceptual situation then arises. For it then turns out that there are three and only three possible geometries compatible with the principle of free mobility—and therefore compatible, as we have seen, with our fundamental coordinating principle linking pure and applied geometry. Our fundamental coordinating principle leaves the choice of Euclidean or non-Euclidean geometry entirely open, and it thus makes perfectly good sense, in this very special conceptual situation, for Poincaré to maintain that the choice of Euclidean geometry is then determined by a convention or stipulation based on its greater mathematical simplicity. In the radically new conceptual situation created by the general theory of relativity, however, this particular view no longer makes sense. Not only is the space-time structure of general relativity incompatible with the fundamental presupposition of the Helmholtz-Lie theory, the principle of free mobility, but, in the general theory, there is only one empirically meaningful way to effect the required coordination between our purely mathematical formulation of the theory (now conceived as a purely formal "conceptual construction") and concrete physical reality—namely, the principle of equivalence, which directly coordinates the

[34] For discussion of free mobility as a coordinating principle for physical geometry see R. DiSalle, "Spacetime Theory as Physical Geometry," *Erkenntnis* 42 (1995), § 2.

purely mathematical notion of a (semi-)Riemannian space-time geodesic with the behavior of freely falling bodies affected only by gravitation. Here, unlike in the very special situation addressed by Poincaré, we are not faced with what we might call a common or generic coordinating principle, which leaves the more specific geometrical structure for physical space still open, but rather with a singular or unique coordinating principle (the principle of equivalence) compatible with one and only one geometrical structure: the geometrical structure for physical space-time described in Einstein's formulation of general relativity.[35] Here the idea of an arbitrary or conventional *choice* of physical geometry has itself lost all real meaning and application—both from a mathematical and an empirical point of view.

By contrast, Einstein's own engagement with the problematic of conventionalism, and, in particular, with the debate on the foundations of geometry between Helmholtz and Poincaré, was an especially timely and fruitful one. It allowed him, as we have seen, to take the critical step, via the principle of equivalence, from the interpretation of three dimensional, non-Euclidean spatial geometry to that of four dimensional, non-Euclidean space-time geometry. And it is in precisely this sense, we might say, that Einstein himself made the crucial transition from nineteenth to twentieth century philosophy of physical geometry. One unforeseen consequence, however, was that the fundamentally new perspective on the foundations of geometry actually created by Einstein in this way—the idea of geometry as fully a branch of physics—has proved much more difficult to grasp than it otherwise might. In particular, in the characteristically twentieth century philosophy of geometry bequeathed to us by logical empiricism, we remained preoccupied with the problematic of conventionalism and the behavior of rigid bodies long after these had lost all specific relevance to physical theory—where a new concern for space-time geometry and the

[35] In particular, to abandon the principle of equivalence and allow non-zero "universal forces" is to introduce empirically meaningless elements into one's formulation admitting no (univocal) coordination with empirical phenomena (see note 33 above). From the point of view of the Cartan-Trautmann formulation, for example, traditional Newtonian gravitation theory involves an arbitrary choice of flat affine connection plus gravitational potential that is not uniquely determined by the empirical local motions. By contrast, the principle of equivalence itself avoids all such arbitrariness by directly coordinating a non-flat affine connection with the empirical local behavior of freely falling bodies. For further discussion see my *Foundations of Space-Time Theories* (note 33), §§ V.4, VII.2.

(essentially four dimensional) problem of motion can now be seen as the true successor to the late nineteenth century tradition in the mathematical and philosophical foundations of geometry that was subject to a far-reaching and radical transformation in the work of Einstein.

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I would like to conclude by suggesting two more general morals about the relationship between philosophy and the sciences that arise from our survey of this particular episode in the transition between late nineteenth and early twentieth century scientific thought. In the first place, Einstein's engagement with nineteenth century philosophical reflection on the foundations of geometry illustrates an important feature of the role of such reflection during truly revolutionary scientific transitions. In particular, when we move from one scientific conceptual framework to a radically different one there is necessarily an intermediate stage in which we are still in the process of transforming the earlier framework but have not yet clearly articulated the later one. There necessarily comes a point, as it were, when we are operating within neither the one nor the other and are, in fact, caught in a deeply problematic (but nevertheless intensely fruitful) state of inter-paradigmatic conceptual limbo. This is illustrated, in the present case, not only by the circumstance that Einstein first applied the principle of equivalence to what we now conceive as the flat geometrical structure of Minkowski space-time (where, more specifically, there is as yet no four dimensional space-time curvature), but, even more strikingly, by the fact that the preceding philosophical debate on the foundations of geometry was framed by the Helmholtz-Lie theorem and was thereby limited to spaces of constant curvature. Since the whole point of the general theory of relativity, in the end, is to describe gravitation by a four dimensional manifold of variable curvature, there is an important sense, as we have seen, in which the final articulation of the general theory rendered the entire preceding debate irrelevant. Nevertheless, as we have also seen, Einstein's final articulation and elaboration of this theory was essentially mediated by precisely this philosophical debate, without which it is indeed hard to imagine how the application of non-Euclidean geometry in physics could have ever been envisaged as a genuinely live alternative.[36]

This leads me to my second moral. It is tempting, when considering the relationship between mathematics, physics, and philosophy, to adopt a simplified linear model: pure mathematics develops possible structures that may be applied in physics, physics surveys the available structures developed by pure mathematics so as to choose an appropriate one for representing some or another physical phenomenon, and philosophy then reflects on this process so as to develop its own doctrines in scientific epistemology and the philosophy of science. In the present case, such a sequence might run: Riemann formulates the mathematical theory of manifolds, Einstein applies this theory in general relativity, and logical empiricism then initiates twentieth century philosophy of geometry by reflecting on Einstein's achievement. Our discussion, brief and selective as it is, has still shown, I hope, that this simplified linear model is entirely inappropriate. It is impossible, in fact, neatly to separate the contributions of mathematics, physics, and philosophy in this way, and it is impossible, in particular, to arrange their respective contributions in a linear sequence. We are rather faced with a highly complex and intensely non-linear process of development wherein mathematics, physics, and philosophy are mutually interacting, and then interdependently evolving, at every stage.

This process begins, in the present case, when scientific thinkers of the late nineteenth century—principally Riemann, Helmholtz, and Poincaré—formulate new perspectives on the nature of geometry, from both mathematical and philosophical points of view, against the background of the original Kantian attempt to comprehend the indispensable role of specifically Euclidean geometry within the Newtonian theory of universal gravitation. The creation of non-Euclidean geometries within pure mathematics was of course the single most important stimulus to these new developments, but they were also framed, as we have suggested, by philosophical concerns going back to Kant, by the question of what it could now mean to apply non-Euclidean geometry within physical theory, and even, especially in Helmholtz's case, by parallel developments within psycho-physiology. Indeed, it was

[36] See again note 29 above. For further discussion of this idea of inter-paradigmatic conceptual limbo, in the context, once again, of Einstein's development of the theory of relativity, see my *Dynamics of Reason* (note 22), Part Two, §§ 3, 4.

precisely this very complex set of interrelated mathematical, philosophical, and what we might call proto-physical developments which then set the stage for Einstein's creation of the theory of relativity. In particular, when Einstein initially hit upon the idea of an application of non-Euclidean geometry to the theory of gravitation he was entirely ignorant of the Riemannian theory of manifolds. He instead located his theorizing within the quite different nineteenth century tradition in the foundations of geometry associated with the work of Klein, as he delicately positioned himself, as we have said, within the debate between geometrical empiricism and conventionalism in the work of Helmholtz and Poincaré. The new theory Einstein actually arrived at then turned out to be incompatible with the fundamental assumption of this tradition, the principle of free mobility, and it was for precisely this reason, in fact, that the mathematics ultimately needed was the much more general theory of manifolds (including spaces of variable curvature) due to Riemann. But there was no direct conceptual route from Riemann's mathematical work to Einstein's new physical theory—which was instead essentially mediated by the debate between Helmholtz and Poincaré on the precise role and significance of rigid bodies in (three dimensional) physical geometry. Moreover, despite the fact that Einstein's finished theory, in an important sense, rendered the entire preceding debate irrelevant, the very salience of this debate in the process of Einstein's creation of the theory propelled twentieth century philosophy of geometry onto a continuation of the problematic initiated by Helmholtz and Poincaré—a continuation, as I have argued, which then proved to be incoherent. Yet my ultimate concern, finally, is not to drive one more nail into the coffin of geometrical conventionalism but rather to begin to convey a sense of the incredible richness of mutual interaction between physics, philosophy, and the foundations of mathematics for which the transition between late nineteenth and early twentieth century scientific thought provides us with an unsurpassed example.

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