Phase-shift analysis for the differential scattering cross-section by the optical model

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Introduction

Previous analyses (1, 3, 5, 9) of angular distribution of proton scattered from nuclei, based on the optical model, have been presented. These analyses have shown that it is possible to give a fairly good account of the experimental differential cross-section with the results obtained theoretically.

The objective of this paper is the theoretical determination of the optical model parameters for an energy of 10 Mev., also the size and shape of the potentials involved. To accomplish these results, two programs were written in PL/1 for the IBM - 360/40 to compute the scattering cross-section for a potential.

The potential used in this calculation is of the form:

$$V(r) = V_c(r) + V_0 f(r) + i W_0 g(r) + V_{s0} \frac{\lambda^2}{a_p} Y(r) (\hat{l} \cdot \hat{s}) \qquad (1)$$

where

$$V_c(r) = \frac{z z' e^2}{r}, \quad f(r) = \left[1 + \exp\left(\frac{r-R}{a_s}\right)\right]^{-1}, \quad g(r) = \exp\left(-\left[\frac{r-R}{a_g}\right]\right),$$
$$Y(r) = \exp\left(-\left[\frac{r-R}{a_y}\right]\right)/r$$

and $R = r_o A^{1/3} + b$. The term f (r) is the Saxon-Woods potential, g (r) is the Gaussian potential, Y (r) is the Yukawa potential and

$V_c(r)$ is the electrostatic Coulomb potential. Emphasis in the calculations of the differential cross-section were concentrated on aluminium. These computations will be compared with experimental results of Dr. Norton M. Hintz from the University of Minnesota.

Optical model analysis

In the nuclear optical model the interaction between a particle and a nucleus is made up of two parts shown in Eq. (1). The nuclear potential is complex, its real part leads to wave refraction and its imaginary part to wave absorption. The wave properties of the nuclear transmission function are found by solving the Schrodinger equation. The principal properties of the nuclear function are:

a) Penetration through the Coulomb and angular momentum region beyond the nuclear radius.

b) Reflection at the abrupt nuclear surface in the vicinity of the nuclear radius.

c) Resonance within the nuclear potential, dampened by the absorption which takes place in the nuclear interior.

The parameters of an appropiate form of complex potential are determined by search procedure where trial parameters are varied over a restricted range until an optimum fit to scattering data is achieved. With the potential so determined, transmission coefficients and cross-section can be evaluated. The optical potential comprises a real central part V_c (r) to represent the Coulomb interaction, in case of charged particles; a real Saxon-Woods part and an imaginary (surface absorption) part with Gaussian exponential form:

$$V(r) = V_c(r) + V_o f(r) + i W_o g(r) \qquad (2)$$

where

$$f(r) = \left[1 + \exp\left(\frac{r-R}{a_s}\right)\right]^{-1} \qquad (3)$$

and

$$g(r) = \exp\left(-\left[\frac{r-R'}{a_s}\right]^2\right) \quad \text{or} \quad g(r) = \exp\left(-\left[\frac{r-R'}{a_s}\right]\right). \tag{4}$$

Various features of this formalism bear closer examination. First, we note the presence of a Coulomb potential in the case of charged-particle scattering. Second, the Saxon-Woods form has both theoretically and empirically been found to be well suited for the real part of the potential, diminishing exponentially with increasing radial distance and essentially constant within the nucleus in conformity with the saturation property of the nuclear forces. Third, the necessity for an imaginary component in the potential, which is useful at low and intermediate energies at which surface absorption may be deemed to occur, and at higher energies where some admixture of volume absorption begins to play a role. To apply to surface absorption, the form factor of the imaginary potential is chosen to peak in the surface region, and is usually taken to be Gaussian or exponential form. Next, it behooves us to consider possible additional terms that may appear in an optical potential. A spin-orbit term is necessary to take account of a scattering interaction. The spin-orbit force vanishes in the interior region of the nucleus and the corresponding potential must therefore show surface peaking. The Thomas and Fenkel form has universally been adopted for the spin-orbit potential.
It may be written as:

$$V_{so}(r) = (V'_{so} + iW_{so})\ h(r)\ (\hat{l}\cdot\hat{s}) \tag{5}$$

where

$$h(r) = -\frac{\lambda^2}{r}\frac{dG(r)}{dr} \tag{6}$$

and

$$G(r) = \left[1+\exp\left(\frac{r-R'}{a_y}\right)\right]^{-1}, \quad \lambda = \frac{\hbar}{m_\pi c}. \tag{7}$$

Although theoretical considerations suggest the inclusion of an imaginary part of spin-orbit i W_{so} whose sign is opposite to that of the real part V_{so} and whose magnitude is at most about one third of V_{so}; most practical analysis nowadays dispense with it and employ solely a real component V_{so}. The customary procedure

is to establish the parameters of the central potential by fitting differential cross-section data, and then to adjust V_{so} until a good fit is obtained.

Mathematical procedure

To perform a scattering experiment, one generally employs a beam of particles coming, for instance, from an acelerator. That packet travels until it hits a target, deforming itself as time goes on, since the particles do not have quite the same velocity.

Theoretically, we know that the wave-function associated with the initial particles obey the free-particle Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\hat{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\hat{r},t) \tag{8}$$

where $\Psi(\hat{r},t)$ is in fact a superposition of plane waves. The target, in which the incident particles collides, usually consist of a macroscopic sample of many individuals scattering centers. Nevertheless, these centers can be assumed to be far enough apart so that, in general, one may neglect coherence effects and confine one's interest to scattering caused by one typical single scattering center. At the point where the particles hit the target, we have to take into account the interaction between the particle and the target by introducing a potential so that the Schrödinger equation is now given by:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\hat{r},t) + V(r)\Psi(\hat{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\hat{r},t). \tag{9}$$

For values of r corresponding to points outside the target, such that the nuclear potential is zero, $\Psi(\hat{r})$ will satisfy equation (8). We now wish to find a particular solution of the Schrödinger equation (9) selected in such a way that it obeys the asymptotic boundary condition

$$\Psi(\hat{r}) = e^{ikz} + A(\theta,\phi)\frac{e^{ikr}}{r} \tag{10}$$

The ultimate goal of the scattering problem is to determine from $\Psi(\hat{r})$ the exact form of the scattering amplitude A (θ $\varnothing$) since it gives the scattering cross-section:

$$\sigma(\theta,\phi) = |A(\theta,\phi)|^2 = A(\theta,\phi)\,A^*(\theta,\phi). \qquad (11)$$

As we saw, our problem will be to find the scattering amplitude for the potentials.

As we shall see, the Coulomb field affects a particle even at large distances in such a way that the wave function is not a plane wave at large distances and the parabolic coordinates are suitable for describing the wave distorted by the Coulomb field. On account of the axial symmetry, we shall have no dependence on the angle $\varnothing$ of the wavefunction describing scattering.

The wavefunction $\Psi_c(\hat{r})$ which describes the scattering process of a particle of reduced mass m by the Coulomb potential satisfies the equation

$$\left(\nabla^2 - \frac{2k\gamma}{r} + k^2\right)\Psi_c(\hat{r}) = 0 \qquad (12)$$

where $K^2 = \frac{2mE}{\hbar^2}$ and $\gamma = \frac{zZe^2}{\hbar v}$. Equation (12) can be solved by using the substitution

$$\Psi_c(\hat{r}) = e^{ikz} f(\hat{r})$$

and expressing the resulting equation for $f(\hat{r})$ in terms of parabolic coordinates (u, v, $\varnothing$) as:

$$\left[\frac{4}{u+v}\left(\frac{\partial}{\partial u}\left(u\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(v\frac{\partial}{\partial v}\right)\right) - \frac{2k\gamma}{u+v} + k^2\right] e^{ikz} f(\hat{r}) = 0. \qquad (13)$$

Solving the above equation we get

$$\Psi_c(\hat{r}) = C\, e^{i(\hat{k}\cdot\hat{r})}\, {}_1F_1\left(-i\gamma, 1; i(kr - \hat{k}\cdot\hat{r})\right) \qquad (14)$$

in which ${}_1F_1$ is the confluent hipergeometric function. The solutions which asymptotically represent an outgoing scattered wave can be obtained by the expansion

$${}_1F_1(a,b;\omega) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-\omega)^{-a}\, {}_2F_0\left(a, a+1-b; -\frac{1}{\omega}\right) + \frac{\Gamma(b)}{\Gamma(a)} e^{\omega}\, \omega^{a-b}\, {}_2F_0\left(1-a, b-a; \frac{1}{\omega}\right)$$

where

$${}_2F_0(s,t;\tfrac{1}{\omega}) \sim \sum_{k=0}^{n} \frac{(s)_k (t)_k}{k!\,\omega^k} = 1 + \frac{st}{\omega(1!)} + \frac{s(s+1)t(t+1)}{\omega^2(2!)} + \dots$$

With these recurrence relations we get the regular asymptotic solution for equation (14)

$$\Psi_c(\hat{r}) \sim \frac{C e^{\frac{\pi}{2}\gamma}}{\Gamma(1+i\gamma)}\left[e^{i(kz + \gamma \ln k(r-z)} + \frac{\gamma}{2k\,\mathrm{sen}^2\frac{\theta}{2}}\, e^{i(\pi + 2\eta_0 - \gamma \ln\frac{1}{2}(1-\cos\theta) - \gamma \ln 2kr + kr)}\frac{e}{r}\right]$$

where the scattering amplitude is

$$A_c(\theta) = \frac{\gamma}{2k\,\mathrm{sen}^2\frac{\theta}{2}}\, e^{i(\pi + 2\eta_0 - \gamma \ln\frac{1}{2}(1-\cos\theta))}$$

We now determine the scattered wave produced by an incident wave of two pure spin states α and β by a modified potential of the form:

$$V'(r) = V_m(r) + V_{so}\,\lambda^2\, V(r)/a_y\ (\hat{l}\cdot\hat{s})$$

where $V_m(r) = V_o f(r) + i\, W_o g(r)$. These states comes from the splitting of the Hamiltonian in two equations separately with the eingenvalues of the operator $\hat{l}\cdot\hat{s}$ as:

$$\hat{H}_+ U_l^{(a)} = \left[\frac{\hat{p}_r^2}{2m} + \frac{l(l+1)\hbar^2}{2mr^2} + V_m(r) + \tfrac{l}{2}\hbar f(r)\right] U_l^{(a)} = 0$$

and

$$\hat{H}\, U_l^{(b)} = \left[\frac{\hat{p}_r^2}{2m} + \frac{l(l+1)\hbar^2}{2mr^2} + V_m(r) - \frac{l+1}{2}\hbar f(r)\right] U_l^{(b)} = 0.$$

For this two different - states we find two regulars solutions requiring that

$$\Psi_{\pm\frac{1}{2}}(\hat{r}) \xrightarrow[r\to\infty]{} \Phi_{\pm\frac{1}{2}}(\hat{r}) + \Psi_{sc}(\hat{r})$$

be satisfied. The asymptotic radial wavefunction with the resulting normalization constant are:

$$U_l^{(a,b)}(r) \xrightarrow[r\to\infty]{} e^{i\delta_l^{(a,b)}} \frac{F_l}{\rho}\cos\delta_l^{(a,b)} + e^{i\delta_l^{(a,b)}}\,\mathrm{sen}\,\delta_l^{(a,b)}\, e^{i(kr + 2\eta_0 - \gamma \ln 2kr - \frac{\pi}{2}l)}$$

where

. $F_\ell \sim e^{i\eta_\ell} \operatorname{sen}(kr - \tfrac{\pi}{2}\ell - \gamma \ln 2kr + \eta_\ell)$, (regular Coulomb function)

$\eta_\ell = \arg\Gamma(\ell+1+i\gamma)$, and δ_ℓ is the phase - shift. The subscripts a and b correspond to each eigenvalue of the operator $\hat{l}.\hat{s}$ in the Hamiltonian respectively.

For the full wavefunction with spin - up we find finally

$$\Psi_{\frac{1}{2}}(\hat{r}) = \Psi_c(\hat{r})\alpha + \frac{e^{i(kr-\gamma\ln 2kr)}}{r}\left[f_{++}\alpha + f_{-+}\beta\right]$$

where

$$f_{++}(\theta,\phi) = \frac{1}{k}\sum_{\ell=0}^{\infty}\left[(\ell+1)e^{i(\delta_\ell^{(a)}+2\eta_\ell)}\operatorname{sen}\delta_\ell^{(a)} + \ell\, e^{i(\delta_\ell^{(b)}+2\eta_\ell)}\operatorname{sen}\delta_\ell^{(b)}\right]P_{\ell 0}(\cos\theta)$$

and

$$f_{-+}(\theta,\phi) = -\frac{1}{2ik}\sum_{\ell=0}^{\infty}\left[e^{i(2\delta_\ell^{(a)}+2\eta_\ell)} - e^{i(2\delta_\ell^{(b)}+2\eta_\ell)}\right]P_{\ell,1}(\cos\theta)\,e^{i\phi}$$

An analogous computation for the spin down yields

$$\Psi_{-\frac{1}{2}}(\hat{r}) = \Psi_c(\hat{r})\beta + \frac{e^{i(kr-\gamma\ln 2kr)}}{r}\left[f_{+-}(\theta,\phi)\,\alpha + f_{--}(\theta,\phi)\,\beta\right]$$

where

$$f_{--}(\theta,\phi) = f_{++}(\theta,\phi) = A_+(\theta)$$

and

$$f_{+-}(\theta,\phi)\,e^{-i\phi} = -f_{+-}(\theta,\phi)\,e^{i\phi} = A_-(\theta)$$

and the total differential scattering cross-section (independent of ϕ) will be

$$\sigma(\theta) = |A_+(\theta)|^2 + |A_-(\theta)|^2 + |A_c(\theta)|^2$$

If a potential has a finite range R as if often the case at least can be simplified considerably. For this case we can split the range of the variable r into two regions; the inner region $r < R$ where $V^1(r)$ exist and the outer region $r > R$ where only the Coulomb potential exist. The problem is now reduced to solving the radial

equations for inner region for which the solutions to Schrodinger equations (15) vanishes at the origin. We must match this inner solution smoothly with the outer solutions of the Coulomb radial wave at r = R. If we assume that we succed in finding an inner solution $u_\ell^{(a,b)}$, mathcing means the numerical identify of the logarithmic derivatives of $u_\ell^{(a,b)}$ at r = R with

$$\Phi_\ell = e^{i\delta_\ell}\left(F_\ell\cos\delta_\ell + G_\ell\operatorname{sen}\delta_\ell\right)$$

where F_l and G_l are the regular and irregular Coulomb functions respectively. The irregular solution has the form

$$G_\ell \sim e^{i\eta_\ell}\cos(kr - \tfrac{\pi}{2}\ell - \gamma\ln 2kr + \eta_\ell)$$

Computing the logarithmic derivatives of Φ_l and $u_\ell^{(a,b)}$ we obtain the fitting conditions

$$S_\ell^{(a)} = e^{i\delta_\ell^{(a)}}\operatorname{sen}\delta_\ell^{(a)} = \frac{F_\ell' u_\ell^{(a)} - u_\ell^{(a)\prime} F_\ell}{u_\ell^{(a)\prime} G_\ell - G_\ell' u_\ell^{(a)} + i\left(u_\ell^{(a)\prime} F_\ell - F_\ell' u_\ell^{(a)}\right)}$$

and (16)

$$S_\ell^{(b)} = e^{i\delta_\ell^{(b)}}\operatorname{sen}\delta_\ell^{(b)} = \frac{F_\ell' u_\ell^{(b)} - u_\ell^{(b)\prime} F_\ell}{u_\ell^{(b)\prime} G_\ell - G_\ell' u_\ell^{(b)} + i\left(u_\ell^{(b)\prime} F_\ell - F_\ell' u_\ell^{(b)}\right)}$$

Conclusion

The level of agreement achieved is illustrated in Figure 1 and 2, and compared with the Hintz's observed scaltering in Figure 3. Figure 2 shows the theoretical curve with spin-orbit interaction and a good agreement is found between this theoretical curve and the experimental results for angular distribution between 30° and 100° and for the shape of the curve. The maximum of the experimental and theoretical curve deviates by a factor of 10 at the same angle. Figure 1 shows the theoretical curve without spin-orbit interaction and no agreement can be found. This means that the spin-orbit term is stronger and the value of the strength of the spin-orbit term has to be larger to affect the central potential.

There are several adjustable parameters V_o, W_o, r_o, V_{so} and a all of which define the two distorting potentials. These parameters are varied to obtain the best fit to the experimental data and the results are shown in Table 1 on Appendix A. Figure 6, illustrate the experimental data.

Figures 4 and 5 illustrate the effect of the increase in the incident energy. The energy of the incident particle should be large enough to overcome the Coulomb barrier of the target nucleus so that the incoming and outgoing waves are distorted by the nuclear potential. If the energy of the incident particle is below 50 Mev., a potential containing a surface absorption term is sufficient to explain the observed scattering.

ACKNOWLEDGMENT

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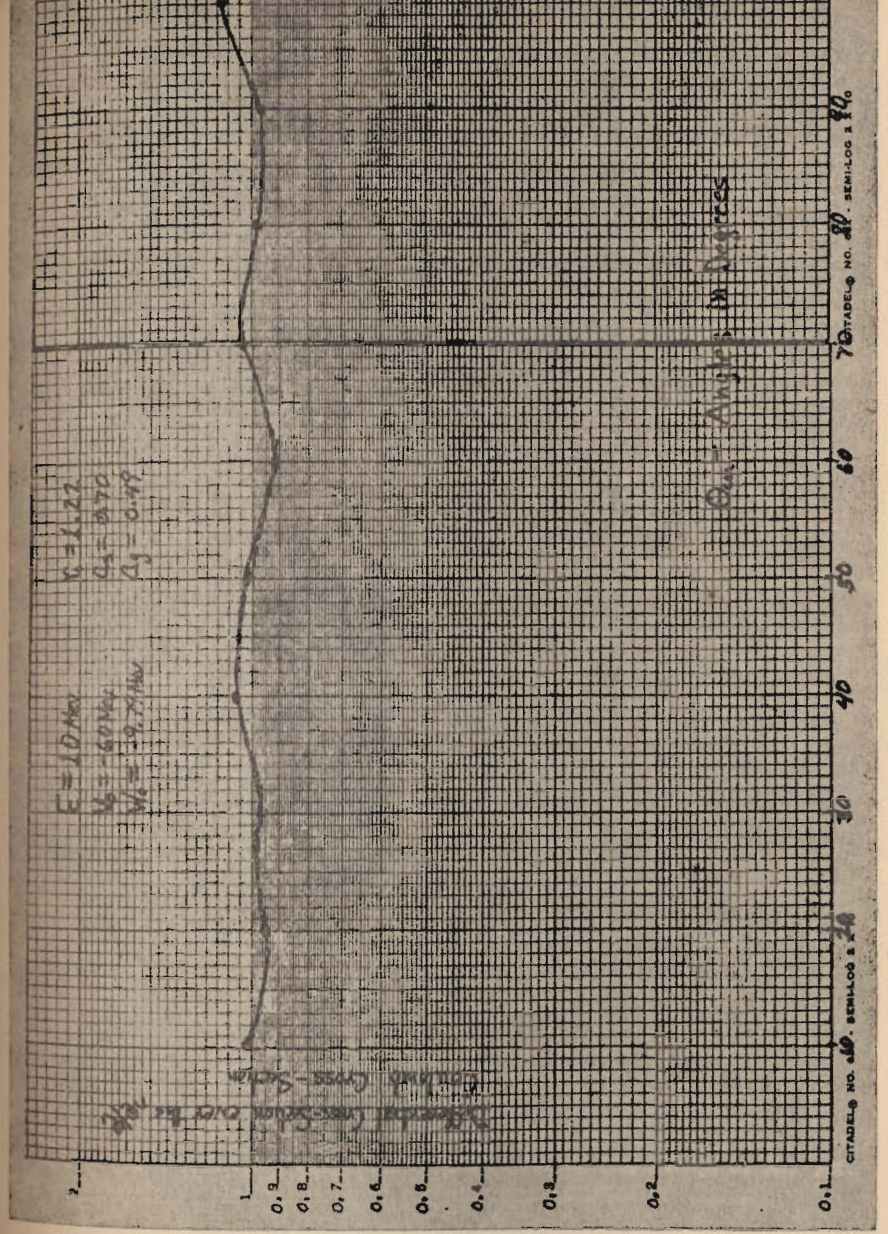

Figure 1.—Theoretical curve for the ratio of the differential scattering cross-section to Rutherford without the spin-orbit interaction.

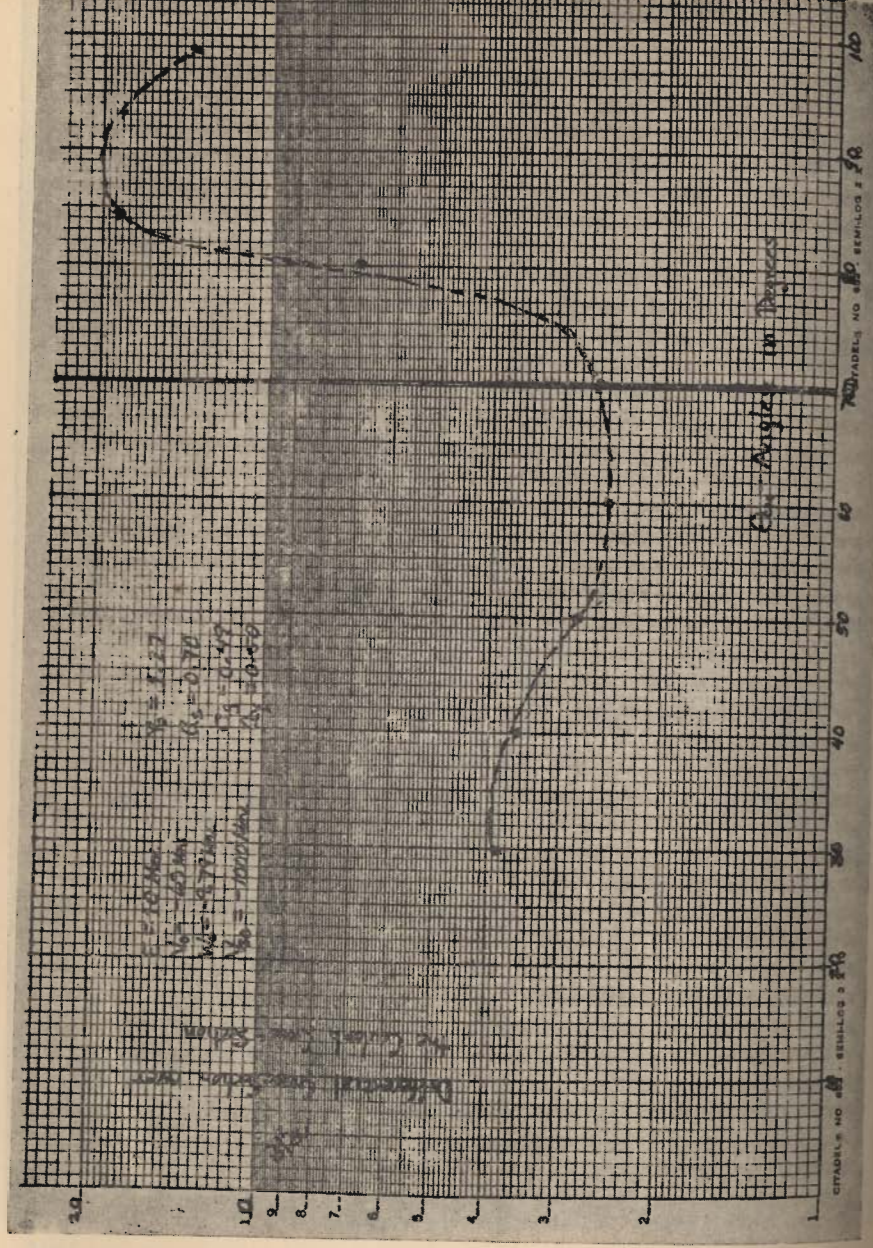

Figure 2. — Theoretical curve for the ratio of the scattering cross-section to Rutherford with spin-orbit interaction.

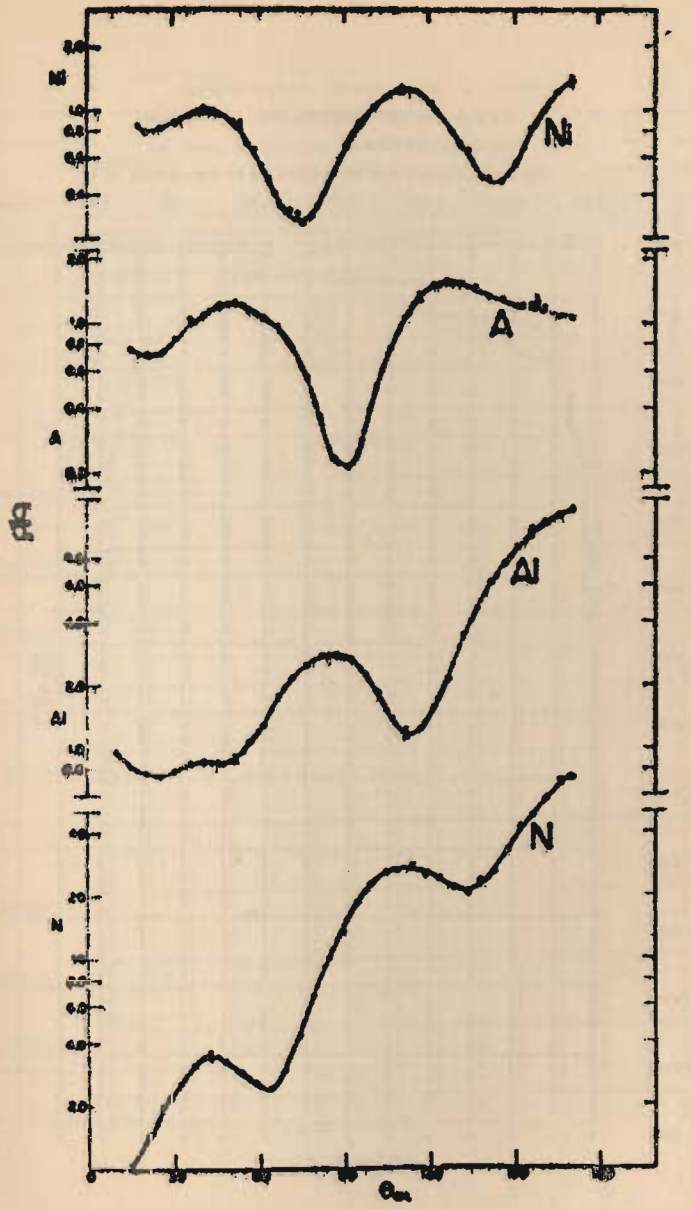

Figure 3. — Hintz's experimental curves for the ratios of scattering differential cross-sections to Rutherford.

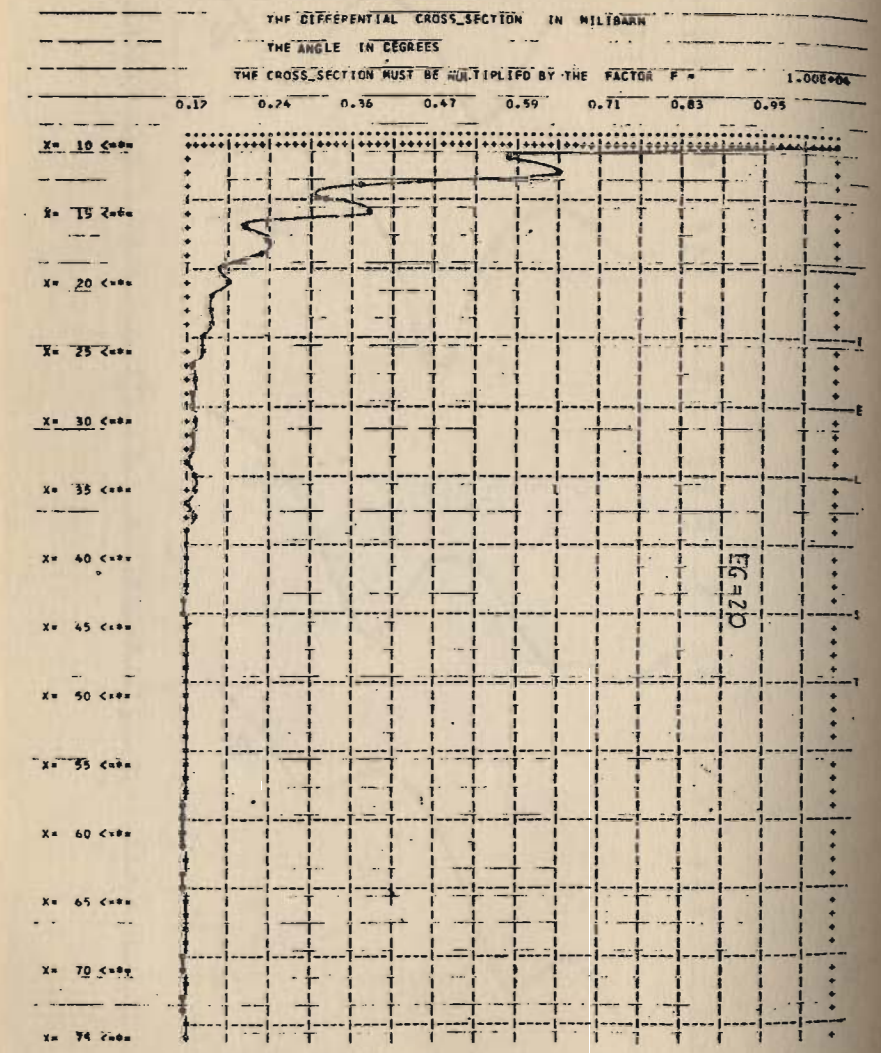

Figure 4.

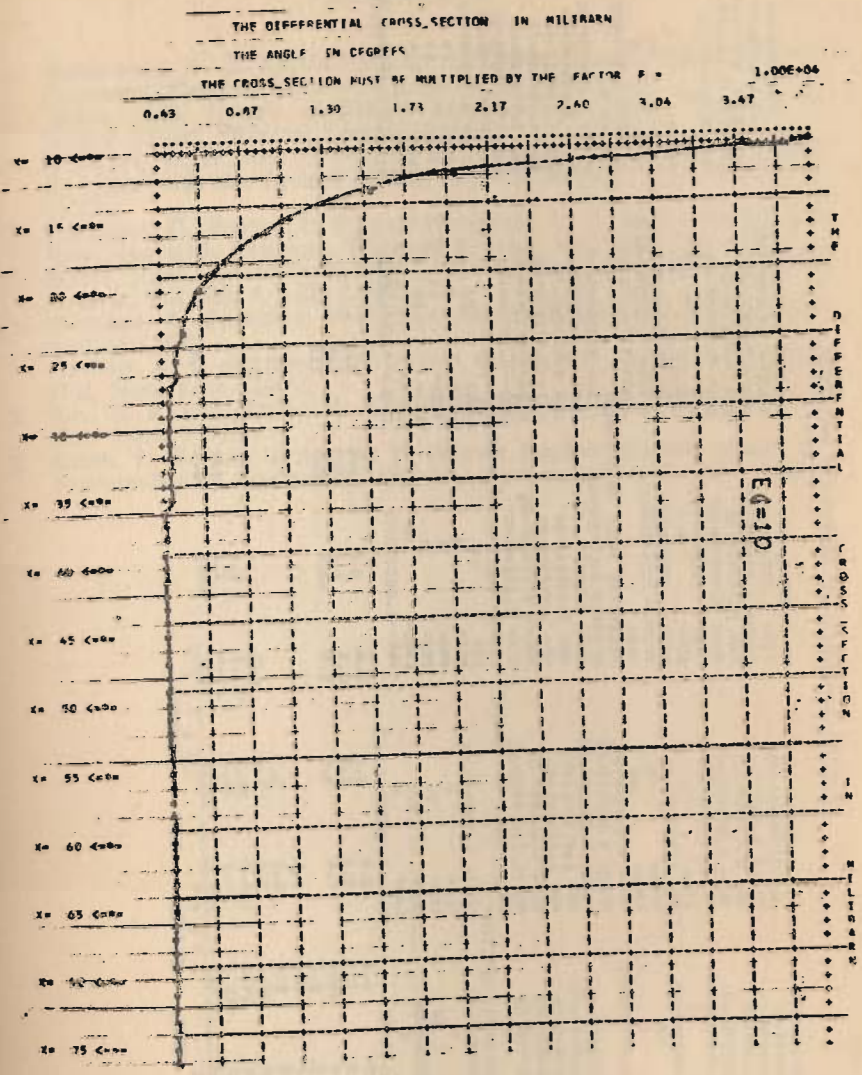

Figure 5.

Figure 6. — Data for computer program.

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RRR= 9.99999E-10         FIVE= 5.000C0E-01        HEAN= 1.05449E+00        HN= 6.62C39E+00          UALFA= 1.37037E+02
OE= 1.6C199E+00          BB= 9.99999F-04          CN= 2.99792F+00          GO= 5.77215E-01          CPOSTH= 3.19999E+00
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COULOMBIAN CROSS-SECTION'                         LC=        24            LOO=        22           PDSW= 6.89999F-01
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        C.0        0.0        0.C        0.0        0.0        0.0        0.0        0.0
       41.0        0.3        0.C        0.0        0.3        C.0        0.3        C.0
        9.0       24.0        6.C        8.0        9,0       48.0       82.0        1.0
        0.0        0.1
        C.0        0.0        0.1
        0.0        0.2       -0.5        0.7
        0.0       -0.6        3.4       -3.0        6.0
        Q.0       24.6      109.0       96.3      -11.3        C.1
        0.0      -16.3       14.1       -9.8       -1.4        1.7        C.1
        0.0        9.3      -25.4       12.2       10.2       -5.5       -0.1        0.9
   T O T A L          C R O S S          S E C T I C N
   TOTAL     CROSS     SECTION     OVER     COULCMB
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APPENDIX A

TABLE 1. *Summary of optical model parameters for proton-nucleus scattering. Energies are in Mev and length in 10^{-13} cm.*

E	V_o	W_o	r_o	R	a_s	a_g	a_y	t_s	t_g	t_y	V_{so}
10	— 60	— 9.799	1.22	3.661	0.70	0.49	0.50	2.24	1.568	1.60	— 1000
20	— 60	— 9.789	1.22	3.661	0.70	0.49	0.50	2.24	1.568	1.60	— 1000
50	— 60	— 9.789	1.22	3.661	0.70	0.49	0.50	2.24	1.569	1.60	— 1000
80	— 60	— 9.789	1.22	3.661	0.70	0.49	0.50	2.24	1.569	1.60	— 1000