Numerical approximation of the motion equation of a simple pendulum, disregarding friction

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Abstract

With the aid of a computer we have found the numerical approximation of the analytical equations of motion of a simple pendulum, disregarding friction, as may be seen in Appendices A, B and C. Appendix D shows the computer program and plot for a pendulum of unrestricted amplitude.

Resumen

Con la ayuda de un computador electrónico, se ha realizado la aproximación numérica de las ecuaciones de movimiento de un péndulo simple, sin considerar rozamientos, tal y como se aprecia en los Apéndices A, B y C. El Apéndice D muestra el programa y representación gráfica de un péndulo que puede oscilar con cualquier amplitud.

A simple pendulum (s.p.) consists of a particle of mass m suspended from a massless string of length l. In the limit of small angles, an s.p. oscillates in simple harmonic motion.

1. *Finding the analytical equation of motion* (angular velocity *vs.* angle).[1]

Our s.p., *Fig. 1,* is a «natural» system in which there are no moving coordinates or constraints. Only conservative forces are acting.

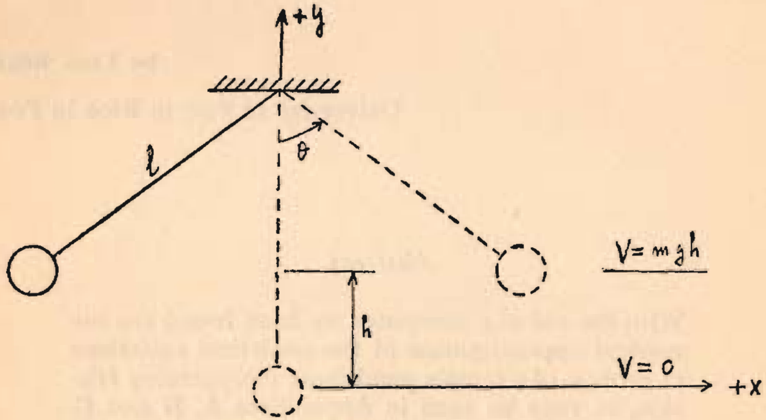

FIGURE 1

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} m (l\dot{\theta})^2 \\ &= \frac{1}{2} m l^2 \dot{\theta}^2. \end{aligned}$$

The potential energy of the mass m is

$$\begin{aligned} V &= mgh \\ &= mg(l - l\cos\theta) \\ &= mgl(1-\cos\theta). \end{aligned}$$

Thus, the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta). \end{aligned}$$

Now

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}.$$

Then Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

gives

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0, \qquad [1]$$

which is the equation of motion of the simple pendulum.

2. *Numerical approximation of the equation of motion, Eq.* [*1*].

i) Euler's Method.[2, 3, 4]

$$\text{Let } \frac{dy}{dx} = f(x, y) \text{ (1st order DE)}$$

and $\dfrac{d^2y}{dx^2} = f'(x, y)$. (2d order DE)

Then if $\dfrac{dy}{dx} = z$,

we have $\dfrac{dz}{dx} = f'(x, y)$.

Thus, if we can find a satisfactory way of solving a 1st order differential equation (DE), we will generally be able to solve higher order DE by reducing them to a system of 1st order DE.

Next, Taylor's series is

$$f(x+\Delta x) = f(x_i) + f'(x_i)\,\Delta x + \frac{f''(x_i)}{2!}\,\Delta x^2 + \dots$$

which we can rewrite in the following way:

$$f_{i+1} = y_i + y_i'\,\Delta x + \frac{y_i''}{2!}\,\Delta x^2 + \dots$$

If we neglect the series after the 2nd term (i.e., Euler's Method) we have

$$y_{i+1} = y_i + y_i'\,\Delta x$$

But,

$$y_i' = f(x_i, y_i)$$
$$y_0' = f(x_0, y_0).$$

Therefore,

$$y_1 = y_0 + y_0'\,\Delta x.$$

This equation is used to evaluate $y_1' = f(x_1, y_1)$, which, in turn, is used to evaluate y_2', and so on.

A very rapid accumulation of error may accompany the use of Euler's Method.

ii) Application of the Euler's method to the equation of motion, Eq. [1].

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \implies \begin{cases} \dfrac{d\theta}{dt} = w \\ \dfrac{dw}{dt} = -\dfrac{g}{l}\sin\theta \end{cases}$$

or

or

$$\left.\begin{aligned} \theta_2 &= \theta_1 + w\,\Delta t \\ w_2 &= w_1 - \frac{g}{l}(\sin\theta)\,\Delta t \end{aligned}\right\} \qquad [2]$$

Suppose the pendulum is allowed to fall from rest (i.e., $t = 0$, $w_o = 0$) at an initial displacement $\theta_o = \dfrac{\pi}{4}$ (= 0.785398). Let $l = 100$ cm. Furthermore, $g = 980$ cm/sec^2.

Now, we would like to print out angular velocity *vs.* angle, plotting values at each 0.1 sec interval. Thus, since 2.0 sec is roughly the period of this pendulum (assuming a small oscillation of, say, less than 15°), we want 20 points printed out. Next we arrange to print out every 50th time increment. We can show this in a diagram:

We will see later a computer program corresponding to this problem (see Section 4, Part (i), and Appendix A).

3. *Finding the analytical equation of motion* (angular momentum *vs.* angle).

In this case we will study the relation of angular momentum *vs.* angle. Thus we have to look for the Hamiltonian equations.[5]

Our simple pendulum is a «natural» system in which there are no moving coordinates or constraints (the time does not enter in the «reduced» transformation equation).[6]

Hence the Hamiltonian H is the total energy of the system (which is constant); i.e.,

$$H \equiv E = T + V$$

or

$$H = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \,(1 - \cos\theta), \qquad [3]$$

where

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

and

$$V = m g l \,(1 - \cos\theta)$$

Now

$$\vec{p}_\theta = m \,(\vec{l} \times \vec{v})$$

and

$$\vec{l} \perp \vec{v}.$$

Then

$$p_\theta = m l v$$

or

$$p_\theta = m l^2 \dot{\theta} \quad \Longrightarrow \quad p_\theta^2 = m^2 l^4 \dot{\theta}^2.$$

That is, Eq. [3] can be written

$$H = \frac{p_\theta^2}{2ml^2} + m g l \,(1 - \cos\theta) \,. \qquad [4]$$

In the s.p. system only conservative forces are acting. Hence, we can write the Hamiltonian equations

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta$$
$$\frac{\partial H}{\partial p_\theta} = \dot{\theta}$$

Therefore

$$\dot{p}_\theta = m g l \sin\theta$$
$$\dot{\theta} = \frac{p_\theta}{m l^2}$$

or

$$\left.\begin{aligned} \dot{p}_\theta &= - m g l \sin\theta \\ \dot{\theta} &= \frac{p_\theta}{m l^2} \end{aligned}\right\} \qquad [5]$$

These are the equations of motion of the simple pendulum.

Before we start the numerical evaluation of the Eq. [5], we may say something else about Eq. [4].

Solving Eq. [4] for $P\theta$, the equation of the system of points in the phase plane is[7]

$$p_\theta = \left\{ 2ml^2 \left[E - mgl\,(1 - \cos\theta) \right] \right\}^{1/2}$$
$$p_\theta = \left[2ml^2 \left(E - mgl + mgl \cos\theta \right) \right]^{1/2}. \qquad [6]$$

If $E - mgl < mgl$ (i.e., $|E| < 2mgl$), then the physical motion of the system can only occur for $|\theta|$ less than a maxima amplitude θ_0 given by the expression

$$\cos\theta_0 = -\frac{E}{2mgl}$$

Now, assume in this case $-\pi < \theta_0 < \pi$. Then, the situation is equivalent to a particle bound in a potential well, where $V(\theta) = mgl(1 - \cos\theta)$, (see *Fig. 2*).

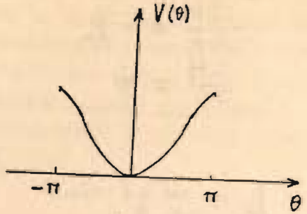

FIGURE 2

The phase space orbits are closed and are given by Eq. [6]. Since the potential is periodic in θ, the orbits are symmetrical with respect to the axes. Furthermore, the points (0,0), (2π, 0), (—2π, 0), are fixed points or points of stable equilibrium (see *Fig. 3*).

Suppose our s.p. is allowed to fall from rest at an initial displacement $-\pi < \theta_o < \pi$. Then we expect a periodic motion (not simple harmonic) with $P\theta = a\ 0$ at $\theta = \pm\theta_o$. If the system is set in motion by a sufficiently large push, however, it will continue to rotate in one direction. This motion will repeat itself periodically, but $P\theta = O$ and θ (t) will continue to increase.[8]

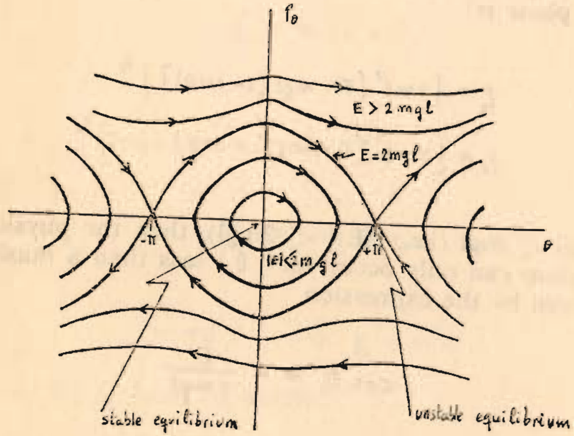

FIGURE 3

If $E - mgl > mgl$ (i.e., $E > 2\ mgl$), θ can increase indefinitely to produce a periodic rotational motion. If $E - mgl = mgl$ (i.e., $E = 2mgl$), the phase paths are cosine functions of the pendulum. In fact, if the pendulum were at rest at $\theta_o = \pi$, then a slight perturbation on the pendulum will slightly change the phase path of its motion. «If the motion were along one of the $E = 2\ mgl$ paths, the pendulum would reach one of the points $\theta = n\pi$ with exactly $P\theta_o = O$, after an *infinite time*»,[9] as is shown in *reference 10.*

4. *Numerical approximation of the equation of motion, Eq.* [5].

i) Euler's Method.
Using the same procedure as in Section 2, Part (ii), we can write

$$D = \Delta t\ (= 0.002\ \text{sec})$$

$$AL = l\ (= 100\ \text{cm})$$

$$AO = \theta_o\ (= 0.785398\ \text{radians})$$

$$PAO = P\theta_o\ (= 0.0)$$

$$G = g\ (= 980\ \text{cm/sec}^2)$$

$$AM = m\ (= 1.0\ \text{gram})$$

$$Y = mgl$$

$$Z = \frac{1}{ml^2}$$

$$A2 = \theta_2\ (= \theta_1 + \frac{P\theta_1}{ml^2}\Delta t)$$

$$PA2 = P\theta_2\ (= P\theta_1 - mgl\ (\sin\theta_1)\ \Delta t).$$

(See flow chart, *Fig. 4*, and the computer program in Appendix A.)

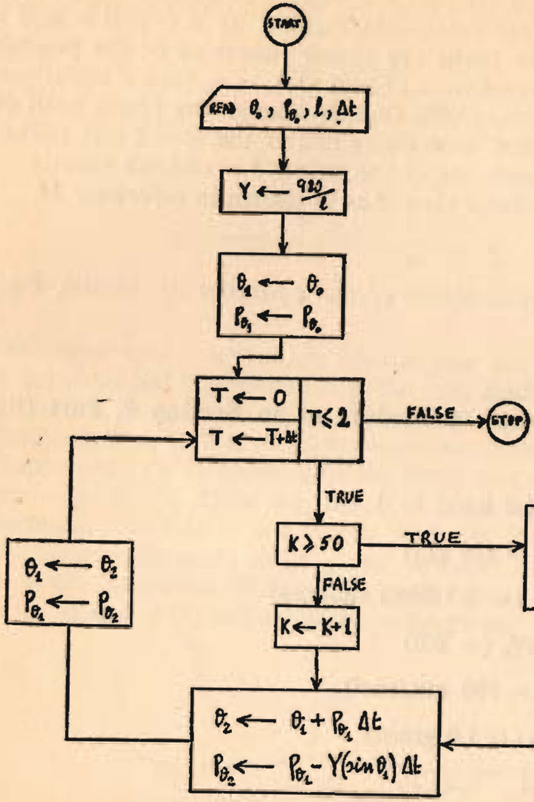

FIGURE 4

ii) Runge Kutta 4th order method.[11, 12, 13]

This is a self-startin gmethod which has less error than the Euler method and the improved Euler method. (Appendix B shows the program to the problem in Part (i) using the Runge Kutta 4th order method.)

Given the equation $y_i' = f(x, y)$, the 4th Runge Kutta solution is

$$y_{i+1} = y_i + \frac{\Delta x}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right],$$

where

$$K_1 = f(x_i, y_i)$$

$$K_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \tfrac{1}{2}K_1\,\Delta x\right)$$

$$K_3 = f\left(x_i + \frac{\Delta x}{2}, y_i + \tfrac{1}{2}K_2\,\Delta x\right)$$

$$K_4 = f(x_i + \Delta x, y_i + K_3\,\Delta x).$$

(See, for example, PECKHAM, *op. cit.*, p. 90.)
The values of K_1, K_2, K_3, and K_4 represent estimates of the slope of the derived function at various points in the interval x_i to x_{i+1}.

For the simple pendulum, the Hamiltonian equations, given in Section 3, are

$$\frac{d\theta}{dt} = \frac{P_\theta}{m l^2}$$

$$\frac{dP_\theta}{dt} = -mgl\sin\theta\,.$$

In our case, $\dot{P}\theta = f(\theta)$, but, more generally, $P\theta = f(t, \theta, \dot{P}\theta)$.

The 4th Runge Kutta solutions are

$$\theta_{i+1} = \theta_i + \frac{\Delta t}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$P_{\theta_{i+1}} = P_{\theta_i} + \frac{\Delta t}{6}\left[l_1 + 2l_2 + 2l_3 + l_4\right],$$

where, assuming that $\dot{P}\theta = f(t, \theta, P\theta)$,

$$K_1 = \theta_i$$
$$l_1 = f(t_i, \theta_i, P_i)$$

$$K_2 = P_{\theta_i} + \frac{\Delta t}{2} l_1$$
$$l_2 = f\left(t_i + \frac{\Delta t}{2}, \theta_i + \frac{\Delta t}{2} K_1, P_{\theta_i} + \frac{\Delta t}{2} l_1\right)$$

$$K_3 = P_{\theta_i} + \frac{\Delta t}{2} l_2$$
$$l_3 = f\left(t_i + \frac{\Delta t}{2}, \theta_i + \frac{\Delta t}{2} K_2, P_{\theta_i} + \frac{\Delta t}{2} l_2\right)$$

$$K_4 = P_{\theta_i} + \Delta t\, l_3$$
$$l_4 = f\left(t_i + \Delta t, \theta_i + \Delta t\, K_3, P_{\theta_i} + \Delta t\, l_3\right).$$

Let us now write down the equivalent nomenclature used in Appendix B:

D = Δt (= 0.0020868 sec)
AL = l (= 100 cm)
G = g (= 980 cm/sec^2)
AM = m (= 1 gram)
AO = θ_o (= 0.785398 radians)
PAO = $P\theta_o$ (= 0.0)
A1 = θ_1
PA1 = $P\theta_1$

$$Z = \frac{1}{ml^2}$$

Y = mgl
B1 = K_1
C1 = l_1
B2 = K_2
C2 = l_2
and so on.

Thus,

```
B1 = Z * PA1
C1 = -Y * SIN (A1)

B2 = Z * (PA1 + (D/2.) * C1)
C2 = -Y * (SIN (A1 + (D/2.) * B1))

B3 = Z * (PA1 + (D/2.) * C2)
C3 = -Y * (SIN (A1 + (D/2.) * B2))

B4 = Z * (PA1 + D * C3)
C4 = -Y * (SIN (A1 + D * B3))

A1 = A1 + (D/6.) * (B1 + 2 * B2 + 2 * B3 + B4)

PA1 = PA1 + (D/6.) * (C1 + 2 * C2 + 2 * C3 + C4).
```

The Runge Kutta method is stable and provides good accuracy.

Appendices C and D show computer programs and plots for a pendulum with different initial angles for each one of the values of the hitting force F = (mgl) E. The hitting force of the program in Appendix C is too small (E = 0.0), so we are dealing with a simple pendulum. Appendix D corresponds to a pendulum of unrestricted amplitude which analytical equation is identical to the «equation of motion for the phase» of the synchrotron principle (see *reference 14*).

5. *Acknowledgment.*

I am grateful to the Department of Computer Sciences of Purdue University, Lafayette, Indiana, U.S.A., which made this work possible.

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APPENDIX A

```
      PROGRAM PENDUL (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C     SIMPLE PENDULUM
C     EULER METHOD
      WRITE(6,6)
      D=.002
      T=0.
      A0=.785398
      PA0=0.
      AL=100.
      G=980.
      AM=1.
      Y=AM*G*AL
      Z=1./(AM*(AL**2))
      A1=A0
      PA1=PA0
      WRITE(6,5)T,A0,PA0
      DO 2 J=1,20
      AJ=J
      T=AJ/10.
      DO 1 K=1,50
      A2=A1+Z*PA1*D
      PA2=PA1-Y*SIN(A1)*D
      A1=A2
      PA1=PA2
    1 CONTINUE
      WRITE(6,5)T,A1,PA1
    2 CONTINUE
    5 FORMAT(10X,F4.1,4X,F11.6,6X,F16.6)
    6 FORMAT(1H1,12X,4HTIME,10X,5HANGLE,7X,16HANGULAR MOMENTUM/)
      END
```

TIME	ANGLE	ANGULAR MOMENTUM
0.0	.785398	0.000000
.1	.751620	-6853.836172
.2	.651255	-13222.489500
.3	.491750	-18567.542486
.4	.286118	-22324.301882
.5	.052743	-24009.272640
.6	-.186193	-23375.746586
.7	-.407654	-20512.854874
.8	-.591051	-15808.815215
.9	-.720578	-9808.259438
1.0	-.786038	-3070.383306
1.1	-.782610	3892.189389
1.2	-.710436	10591.110038
1.3	-.574648	16508.406421
1.4	-.385808	21078.672455
1.5	-.160125	23757.516900
1.6	.081438	24169.193678
1.7	.315593	22248.159788
1.8	.520097	18267.062522
1.9	.876684	12727.664947
2.0	.772572	6201.418122

APPENDIX B

```
      PROGRAM PENDUL (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C     SIMPLE PENDULUM
C     RUNGE KUTTA 4TH ORDER METHOD
      COMMON D02,A1,PA1,Y,Z,D
      WRITE(6,6)
      D=.0020868
C     PERIOD=20*50*D=2.0868 SEC.
      T=0.
      A0=.785398
      PA0=0.
      AL=100.
      G=980.
      AM=1.
      Y=AM*G*AL
      Z=1./(AM*(AL**2))
      A1=A0
      PA1=PA0
      WRITE(6,5)T,A0,PA0
      DO 2 J=1,20
      AJ=J
      T=1.04304*AJ/10.
      CALL RUNGE
      WRITE(6,5)T,A1,PA1
    2 CONTINUE
    5 FORMAT(10X,F8.6,2X,F11.6,4X,F16.6)
    6 FORMAT(1H1,12X,4HTIME,10X,5HANGLE,7X,16HANGULAR MOMENTUM/)
      STOP
      END
```

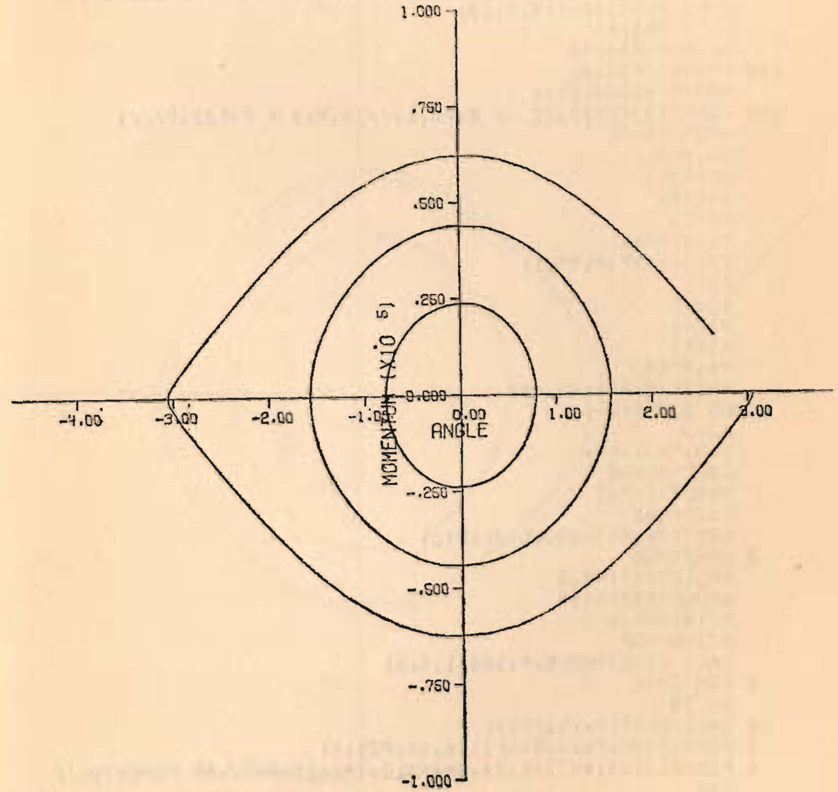
```
      SUBROUTINE RUNGE
      COMMON D02,A1,PA1,Y,Z,D
      DO 10 I=1,50
      D02=D/2.
      B1=Z*PA1
      C1=-Y*SIN(A1)
      B2=Z*(PA1+D02*C1)
      C2=-Y*(SIN(A1+D02*B1))
      B3=Z*(PA1+D02*C2)
      C3=-Y*(SIN(A1+D02*B2))
      B4=Z*(PA1+D*C3)
      C4=-Y*(SIN(A1+D*B3))
      A1=A1+(D/6.)*(B1+2*B2+2*B3+B4)
      PA1=PA1+(D/6.)*(C1+2*C2+2*C3+C4)
   10 CONTINUE
      RETURN
      END
```

TIME	ANGLE	ANGULAR MOMENTUM
0.000000	.785398	0.000000
.104304	.747915	-7138.813245
.208608	.638378	-13714.016672
.312912	.465825	-19111.133197
.417216	.245883	-22696.235253
.521520	.000311	-23959.766012
.625824	-.245293	-22702.429918
.730128	-.465328	-19122.571752
.834432	-.638022	-13729.195900
.938736	-.747730	-7156.140193
1.043040	-.785398	-18.016483
1.147344	-.748101	7121.482831
1.251648	-.638735	13698.830148
1.355952	-.466321	19099.683331
1.460256	-.246473	22690.026007
1.564560	-.000934	23959.750140
1.668864	.244703	22708.609993
1.773168	.464830	19133.998987
1.877472	.637664	13744.367823
1.981776	.747543	7173.463665
2.086080	.785397	36.032958

APPENDIX C

```
      PROGRAM PENDUL (INPUT,OUTPUT,PLOT,TAPE5=INPUT,TAPE6=OUTPUT)
C SIMPLE PENDULUM WITH DIFFERERENT INITIAL ANGLES FOR EACH ONE
C OF THE VALUES OF THE HITTING FORCE = (M*G*L)*E
C     RUNGE KUTTA 4TH ORDER METHOD
      DIMENSION ANGLE(102),P(102)
      COMMON D02,A1,PA1,Y,Z,D,A0,E
C  A0 = INITIAL ANGLE IN RADIANS
      CALL PLOTS
    1 READ(5,103)E
  103 FORMAT(F4.1)
      IF(E.GT.1.)GO TO 10
      PMIN=0.0
      PMAX=100000.
      AMAX=10.
      AMIN=0.0
      DP=PMAX/4.0
      CALL AXIS(0.,5.0,5HANGLE,-5,20.,0.0,-10.0,1.0,0)
      CALL AXIS(10.,1.0,8HMOMENTUM,+8,8.0,90.,-PMAX,DP,-1)
      CALL PLOT(10.0,5.0,-3)
      DO 8 I=1,5
      READ(5,100)A0
  100 FORMAT(F11.8)
      WRITE(6,200)E,A0
  200 FORMAT(1H1,6X,≠E  = ≠,F4.1,//7X,≠A0 = ≠,F10.8///)
      WRITE(6,6)
      D=.001
      AL=100.
      G=980.
      AM=1.
      Y=AM*G*AL
      Z=1./(AM*(AL**2))
      T=0.
      A0=A0
      PA0=0.0
      A1=A0
      PA1=PA0
      WRITE(6,5)T,A0,PA0
      DO 2 J=1,100
      AJ=J
      T=5.*AJ/100.
      CALL RUNGE
      ANGLE(J)=A1
      P(J)=PA1
      WRITE(6,5)T,ANGLE(J),P(J)
    2 CONTINUE
      ANGLE(101)=0.0
      ANGLE(102)=1.0
      P(101)=0.0
      P(102)=DP
      CALL LINE(ANGLE,P,100,1,0,0)
    8 CONTINUE
      GO TO 1
   10 CALL PLOT(0.,0.,999)
    5 FORMAT(8X,F8.2,2X,F11.6,6X,F20.6)
    6 FORMAT(10X,4HTIME,8X,5HANGLE,14X,16HANGULAR MOMENTUM/)
      END
```

```
      SUBROUTINE RUNGE
      COMMON D02,A1,PA1,Y,Z,D,A0,E
      DO 10 I=1,50
      D02=D/2.
      B1=Z*PA1
      C1=-Y*SIN(A1)+Y*E
      B2=Z*(PA1+D02*C1)
      C2=-Y*SIN(A1+D02*B1)+Y*E
      B3=Z*(PA1+D02*C2)
      C3=-Y*SIN(A1+D02*B2)+Y*E
      B4=Z*(PA1+D*C3)
      C4=-Y*SIN(A1+D*B3)+Y*E
      A1=A1+(D/6.)*(B1+2*B2+2*B3+B4)
      PA1=PA1+(D/6.)*(C1+2*C2+2*C3+C4)
   10 CONTINUE
      RETURN
      END
```

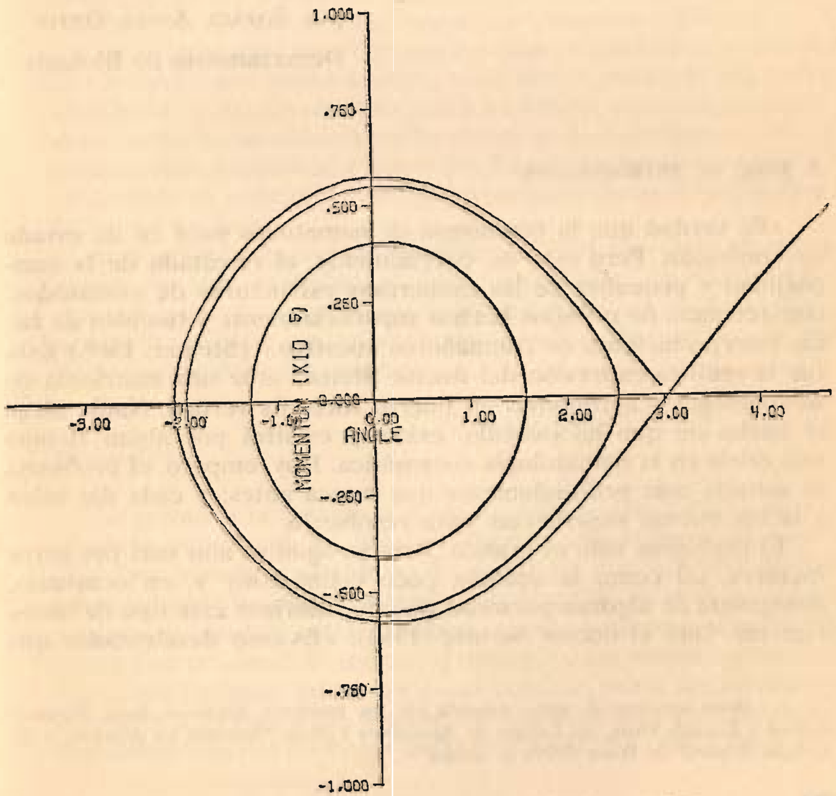


Plot corresponding to the program in Appendix C.

APPENDIX D

```
      PROGRAM PENDUL (INPUT,OUTPUT,PLOT,TAPE5=INPUT,TAPE6=OUTPUT)
C SIMPLE PENDULUM WITH DIFFERERENT INITIAL ANGLES FOR EACH ONE
C OF THE VALUES OF THE HITTING FORCE = (M*G*L)*E
C     RUNGE KUTTA 4TH ORDER METHOD
      DIMENSION ANGLE(102),P(102)
      COMMON DO2,A1,PA1,Y,Z,D,A0,E
C  A0 = INITIAL ANGLE IN RADIANS
      CALL PLOTS
    1 READ(5,103)E
  103 FORMAT(F4.1)
      IF(E.GT.1.)GO TO 10
      PMIN=0.0
      PMAX=100000.
      AMAX=10.0
      AMIN=0.0
      DP=PMAX/4.0
      CALL AXIS(0.,5.0,5HANGLE,-5,20.,0.0,-10.0,1.0,0)
      CALL AXIS(10.0,1.0,8HMOMENTUM,+8,8.0,90.,-PMAX,DP,-1)
      CALL PLOT(10.0,5.0,-3)
      DO 8 I=1,4
      READ(5,100)A0
  100 FORMAT(F11.8)
      WRITE(6,200)E,A0
  200 FORMAT(1H1,6X,≠E  = ≠,F4.1,//7X,≠A0 = ≠,F10.8///)
      WRITE(6,6)
      D=.001
      AL=100.
      G=980.
      AM=1.
      Y=AM*G*AL
      Z=1./(AM*(AL**2))
      T=0.
      A0=A0
      PA0=0.
      A1=A0
      PA1=PA0
      WRITE(6,5)T,A0,PA0
      DO 2 J=1,100
      AJ=J
      T=5.*AJ/100.
      CALL RUNGE
      ANGLE(J)=A1
      P(J)=PA1
      WRITE(6,5)T,ANGLE(J),P(J)
    2 CONTINUE
      ANGLE(101)=0.0
      ANGLE(102)=1.0
      P(101)=0.0
      P(102)=DP
      CALL LINE(ANGLE,P,100,1,0,0)
    8 CONTINUE
      GO TO 1
   10 CALL PLOT(0.,0.,999)
    5 FORMAT(8X,F8.6,2X,F11.6,6X,F20.6)
    6 FORMAT(10X,4HTIME,8X,5HANGLE,14X,16HANGULAR MOMENTUM/)
      END
```

```
      SUBROUTINE RUNGE
      COMMON DO2,A1,PA1,Y,Z,D,A0,E
      DO 10 I=1,50
      DO2=D/2.
      B1=Z*PA1
      C1=-Y*SIN(A1)+Y*E
      B2=Z*(PA1+DO2*C1)
      C2=-Y*SIN(A1+DO2*B1)+Y*E
      B3=Z*(PA1+DO2*C2)
      C3=-Y*SIN(A1+DO2*B2)+Y*E
      B4=Z*(PA1+D*C3)
      C4=-Y*SIN(A1+D*B3)+Y*E
      A1=A1+(D/6.)*(B1+2*B2+2*B3+B4)
      PA1=PA1+(D/6.)*(C1+2*C2+2*C3+C4)
   10 CONTINUE
      RETURN
      END
```



Plot made by computer as programmed in Appendix D.